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ABSTRACT

This course is intended for students who have a thorough knowledge of college preparatory mathematics, including algebra, axiomatic geometry, trigonometry, and analytic geometry. This text, Part I, contains the first five chapters of the course and two appendices. Chapters included are: (1) Polynomial Functions; (2) The Derivative of a Polynomial Function; (3) Circular Functions; (4) Derivatives of Circular Functions; and (5) Exponential and Related Functions. The appendices are: (1) Functions and Their Representations; and (2) Polynomials. (RH)

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SCHOOL
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CALCULUS OF ELEMENTARY FUNCTIONS

Part I

Student Text

(Revised Edition)

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CALCULUS OF ELEMENTARY FUNCTIONS

Part I

Student Text

(Revised Edition)

The following is a list of all those who participated in the preparation of this volume:

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FOREWORD

The correspondence between graphs in the xy-plane and relations between x and y was one of the profound discoveries of mathematics. In particular, if no vertical line can meet a graph in more than one point, then the graph is that of a function $f : x \to y$; that is, for each first coordinate x of a point on the graph there is a unique number y such that (x,y) lies on the graph. A central purpose of this text is to study the relationship between graphs of functions and the expressions which define these functions. We shall concentrate our attention on functions defined by polynomial, trigonometric, exponential and logarithmic expressions, or by combinations of such expressions. These functions are usually referred to as elementary functions.

We should expect that properties of the graph of a function are related to the expression which defines the functions. For example, by analyzing the functional expression we should be able to determine the location of high and low points on the graph, and in addition answer questions about the shape of the graph (such as intervals of rise or fall and how the graph bends). Furthermore, if the function is related to some physical problem involving motion it seems reasonable that an analysis of its expression should enable us to determine such aspects of the motion as velocity and acceleration. Likewise, we might expect to be able to determine from the functional expression such properties as the average value of the function and the area of a region bounded by the function.

Our aim is to develop some of the concepts and techniques which will enable us to obtain important information about graphs of elementary functions. The primary concept which we develop in Chapters 2, 4, and 6 is that of the tangent line at a point on the graph of a function. This tangent line is described as the straight line which best fits the graph near that point. In particular, formulas are developed for finding the slopes of tangent lines to the graphs of various elementary functions. For a given function f, such a formula defines a new function called the derivative of f. Values of the derivative give a measure of the rate of change of the graph. In particular, such aspects of motion as velocity and acceleration are described by the derivative. Furthermore, high and low points of the graph of f can be given by zeros of the derivative, while rise or fall can be determined by the sign of the derivative.

Our first task is to analyze the simplest kinds of elementary functions, namely polynomials, the sine and cosine functions, and simple power, exponential and logarithmic functions. These are discussed in Part One, while the more difficult techniques associated with various algebraic combinations (sums, products, quotients, composition) of these basic functions are left to Chapter 8 (in Part Two). In addition, the concept of antiderivative is introduced in Part Two, the antiderivative of f being a function whose derivative is f. We shall show how antiderivatives can be used to calculate areas and to solve such problems as determining velocity given acceleration. Furthermore, this relationship between antiderivatives and area provides us with a powerful geometric tool for analyzing and approximating functions.

Chapters 1 and 2 discuss polynomial functions, with Chapter 1 concentrating on definitions and simple algebraic and geometric properties. The concept of tangent line at a point on the graph of polynomial function is introduced in Chapter 2 and formulas for the derivative of a polynomial function are obtained. Applications of the derivative to graphing (such as finding high and low points and intervals of rise or fall), its interpretation as velocity or acceleration, and its use in approximation are also discussed in Chapter 2. This same pattern is followed in the remaining four chapters of this volume. Definitions and simple properties of the sine and cosine functions and the power exponential and logarithm functions appear in Chapters 3 and 5, respectively, while Chapters 4 and 6 discuss derivative formulas for these respective classes, as well as applications to graphing and approximations.

For their thoughtful comments we are grateful to Frank B. Allen, Lyons Township High School; Leonard Gillman, University of Texas; David W. Jonah, Wayne State University; Albert W. Tucker, Princeton University; and the pilot teachers and students at Cubberley, Gunn, and Palo Alto High Schools in Palo Alto, California; St. Mark's School in Dallas, Texas; and Simon's Rock in Great Barrington, Massachusetts.

We are also indebted to previous SMSG writing teams, whose materials had considerable influence on this text. Many ideas and exercises were taken directly or adapted from two earlier SMSG texts: *Elementary Functions* and *Calculus*. In the first part of this text we borrow heavily from SMSG *Elementary Functions*; Appendix 2 of this text contains Sections 2-12, 2-6, 2-8, and 2-9 of *Elementary Functions*. Appendices 1, 3, 4, 5, 6, 7, and 8 were adapted from Sections A2-1, 2, 3, 4, 6; A3-1, 2; 10-3, 4, 5, 6; 6-1, 2, 3, 4; A6-1, 2; 3-2, 3, 4,; A4-1, 2; 5-3; and A1-5 of SMSG *Calculus*.

Finally, we express appreciation to Nancy Woodman, whose function as a typist was transcendental.

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Chapter 1

POLYNOMIAL FUNCTIONS

This is the first of two chapters on polynomial functions. Here we concentrate on the definition and simple algebraic and geometric properties of polynomial functions, leaving to the next chapter a discussion of tangents to polynomial graphs.

Our concern is with the relationship between the graph of a polynomial function and the expression which defines the function. This relationship is easy to describe for constant and linear functions for these correspond to nonvertical lines (Section 1-2). For quadratic functions the graph is a parabola whose location and general slope can be easily determined by using the quadratic formula. In fact any quadratic graph is just a translation or scale change of the graph of the squaring function (Section 1-3).

After discussing these familiar cases we turn to polynomial functions of degree three or larger. Here the situation becomes less routine. Synthetic division serves initially as a tool for plotting points (Section 1-4). Later interpretations (Chapter 2) are more profound. The general relation between zeros and factors of the polynomial is discussed in Section 1-5. In Sections 1-6, 1-7, and 1-8 we discuss methods for locating, determining, and approximating zeros of polynomial functions. The final section of this chapter indicates some of the kinds of information about the graph of a polynomial function which can be quickly obtained from its expression, including the important result that the degree is a bound for the number of times its graph crosses the x-axis. Further algebraic results are discussed in Appendix 2.

We begin our study with polynomial functions because they are the simplest of the elementary functions. The theory and techniques employed in Chapters 1 and 2 are fundamental to the rest of the text. Not only will our approach to the analysis of polynomial functions be useful as we deal with other functions, but the polynomial functions themselves will serve as approximations to other functions we shall study.

1-1. Introduction and Notation

In this chapter we shall be concerned with functions that are defined by expressions of the form

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

where n is a non-negative integer and the coefficients $a_i (i = 0, 1, 2, 3, \dots, n)$ are real numbers. Such expressions are called polynomials, and the functions they define are called polynomial functions.

We commonly denote functions by a single letter f, using the symbol $f(x)$ to denote the value of f at the point x. Thus a polynomial function f is a function whose rule is given by

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n.$$

This notation is particularly useful when we wish to calculate various values of f. For example, suppose f is given by

$$f(x) = 2 + x - x^2. \tag{1}$$

The values $f(0)$, $f(-1)$ and $f(\frac{1}{2})$ are then given by

$$f(0) = 2 + 0 - 0^2 = 2$$
$$f(-1) = 2 + (-1) - (-1)^2 = 0$$
$$f(\tfrac{1}{2}) = 2 + \tfrac{1}{2} - (\tfrac{1}{2})^2 = 2\tfrac{1}{4}.$$

We can substitute other letters or expressions for x; for example

$$\begin{aligned} f(t) &= 2 + t - t^2, \\ f(2 - y) &= 2 + (2 - y) - (2 - y)^2 \\ &= 3y - y^2, \\ f(a + b) &= 2 + (a + b) - (a + b)^2 \\ &= 2 + a + b - a^2 - 2ab - b^2. \end{aligned}$$

We may also denote the function defined in (1) by

$$f : x \to 2 + x - x^2, \tag{2}$$

thus stressing that f is an operation or association. We frequently introduce another variable to stand for $f(x)$. (This is especially convenient for graphing.) For example we may rewrite (2) as $f : x \to y$, where

$$y = 2 + x - x^2. \tag{3}$$

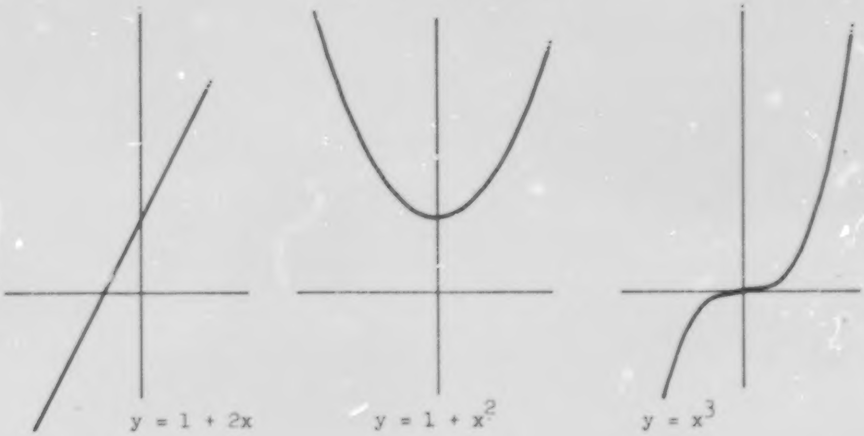


Figure 1-1a

The graph of a function f is the set of pairs $(x,f(x))$ as we picture them on a plane, say the xy-plane. (In Figure 1-1a we sketch the graphs of three polynomial functions.) Much of our effort in this and the next chapter will be directed toward quickly obtaining such pictures. The graph (a model) can help us to examine the behavior of a function (which may itself be an idealized mathematical model of some physical situation). Polynomial functions often arise in applications. We give here two examples.

Example 1-1a. If we say "the volume of a sphere is a function of its radius" we mean that if f is the volume function and r is the measure of the radius then $f : r \to V$, where V is the measure of the volume. In particular we know that

$$V = \frac{4}{3}\pi r^3. \tag{4}$$

The expression $\frac{4}{3}\pi r^3$ is, of course, defined for any real number r. The volume function $r \to V$ is, however, defined only when $r \geq 0$.

If the radius is doubled we can write

$$f(2r) = \frac{4}{3}\pi(2r)^3 = \frac{32}{3}\pi r^3 = 8(\frac{4}{3}\pi r^3),$$

which tells us that doubling the radius multiplies the volume by eight.

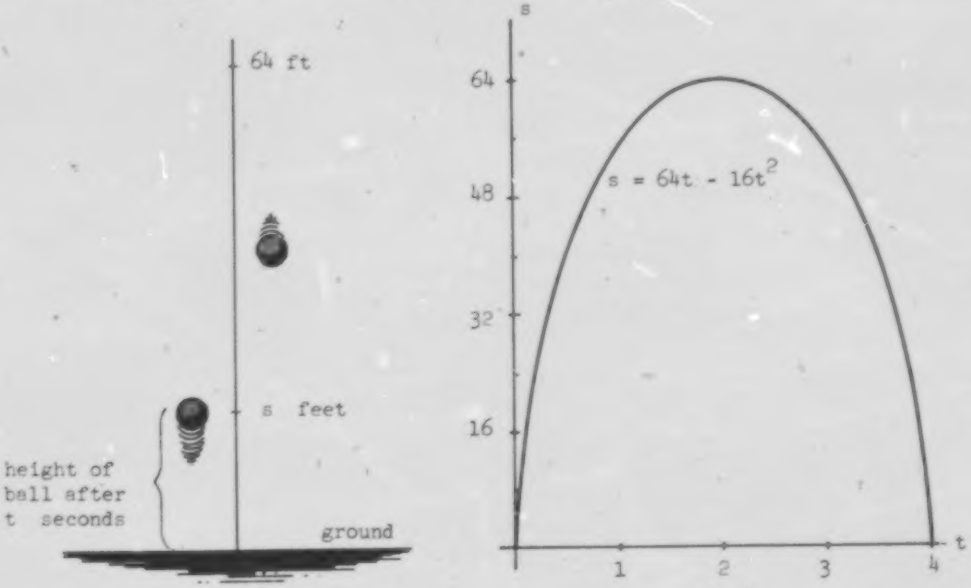
Example 1-1b. A ball is thrown straight up with an initial velocity of 64 feet per second so that its height s feet above the ground after t seconds is given by the function

$$(5) \qquad t \to s = 64t - 16t^2.$$

(We shall later derive functions such as this as a consequence of various physical assumptions about velocity, acceleration and gravity.) This function can only serve as an idealized model of the physical situation over a particular interval of values for t. Since $s = 0$ when $t = 0$ or $t = 4$, the function serves as a mathematical model over the interval $0 \leq t \leq 4$; the ball is in the air for 4 seconds. To find how many seconds it takes for the ball to reach its maximum height we can (by completing the square) write the function in the form

$$t \to -16(t - 2)^2 + 64.$$

The quantity $-16(t - 2)^2$ is negative unless $t = 2$. Thus s cannot exceed 64 and equals 64 only when $t = 2$. Therefore, we conclude that the ball reaches a maximum height of 64 feet after 2 seconds.



(i) (ii)

Figure 1-1b

While we picture (Figure 1-1b) the motion function (5) as a parabola (ii), we think of the physical motion of the ball itself as vertical (i).

Exercises 1-1

1. In Example 1-1a we expressed the volume of a sphere as "a function of its radius." Express the volume of a sphere as "a function of its diameter."

2. Suppose that a pellet is projected straight up and after a while comes straight down via the same vertical path to the place on the ground from which it was launched. After t seconds the distance s feet of the pellet above the ground is described by the equation

$$s = 160t - 16t^2,$$

which defines the function

$$f : t \rightarrow 160t - 16t^2.$$

(a) What is the value of s when $t = 4$?

(b) Evaluate $f(6)$.

(c) How high above the ground is the pellet after 4 seconds?

(d) What is the height of the pellet after 6 seconds?

(e) Compare your answers for parts (c) and (d). Explain on physical grounds.

(f) How many seconds is the pellet in the air?

(g) How long does it take the pellet to reach its highest point?

(h) How high does the pellet go?

3. Tom is standing on the top of a railroad car which is moving at a speed of 32 ft./sec. as it passes a station. As he passes Dick on the station platform Tom throws a ball straight upward with an initial speed of 64 ft./sec. After t seconds the ball is a horizontal distance of x feet and a vertical distance of y feet from a point opposite Dick. The distances x ft. and y ft. are given by the equations

$$x = 32t$$

and

$$y = 64t - 16t^2.$$

(a) Does Tom have to move to catch the ball?

(b) What is the path of the ball as Dick sees it from the platform?

(c) Write y in terms of x.

(d) Name the curve that is the graph of your equation in part (c).

(e) Sketch the graph of $y = 2x - \frac{x^2}{64}$.

(f) For what values of x does $y = 0$?

(g) What is t when $x = 128$?

(h) After how many seconds does Tom catch the ball?

(i) How far down the platform from Dick does Tom catch the ball?

1-2. Constant and Linear Functions

The simplest polynomial functions are constant functions. If c is any real number, then the function f which associates with every real number x the value c,

$$f : x \to c,$$

is called a constant function. The graph of such a function is a line parallel to and $|c|$ units from the x-axis. Some examples of constant functions are graphed in Figure 1-2a.

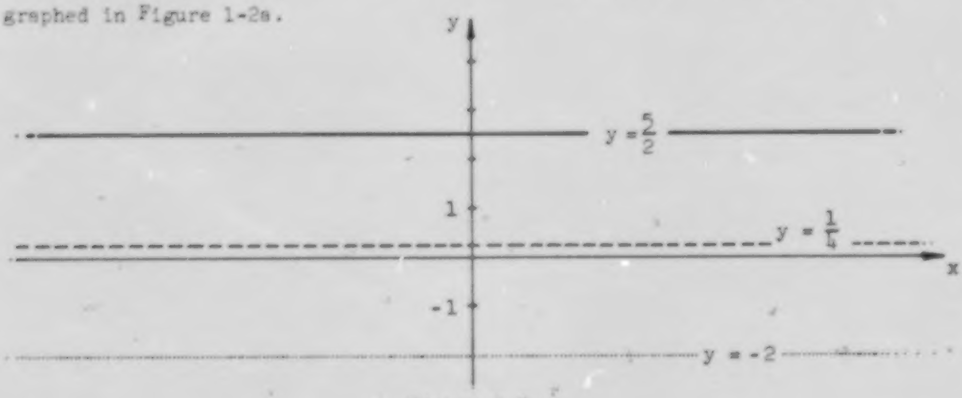


Figure 1-2a

Constant functions are quite simple, yet they occur frequently in mathematics and science. A physical example of such a function is

$$f : t \to 32.$$

Here the constant is the acceleration due to gravity, that is, the constant amount by which the velocity of fall increases each second. When distance is measured in feet and time in seconds, this constant is very nearly 32 at sea level. In other words, the velocity of a falling body increases 32 feet per second every second.

A simple principle will later be useful when we encounter constant functions:

(1) if f is a constant function and the value $f(a)$ is known, then $f(x) = f(a)$ for all x.

For example, if we know that f is a constant function and that $f(0) = 10$, then we know that $f(3)$ is also 10. Clearly, (1) is just a restatement of the fact that if f is a constant function then all values of f are the same.

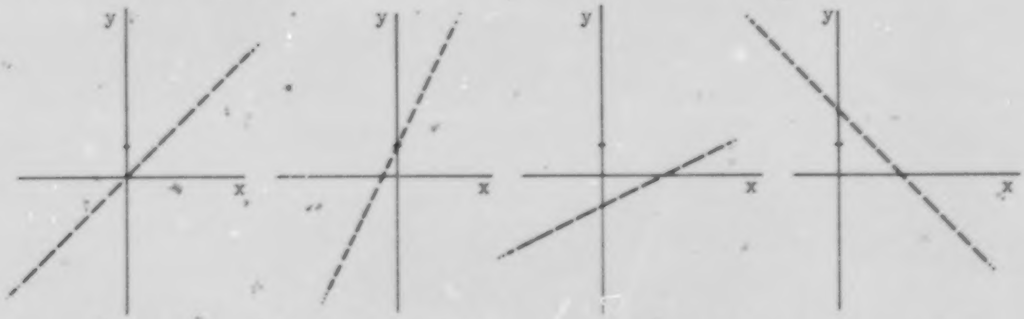
A linear function f is a function defined by an expression of the form mx + b; that is, f is linear if there are numbers m and b such that for all x

$$f(x) = mx + b, \quad m \neq 0. \tag{2}$$

If m = 0 then f is a constant function. The graph of (2) is a line which crosses the y-axis at the point (0,b), since

$$f(0) = b;$$

b is called the y-intercept of f. The number m is called the slope of f, and gives a measure of the steepness of the graph. Four linear functions are sketched in Figure 1-2b.



$x \rightarrow y = x$ $\qquad$ $x \rightarrow y = 2x + 1$ $\qquad$ $x \rightarrow y = \frac{1}{2}x - 1$ $\qquad$ $x \rightarrow y = -x + 2$

Figure 1-2b

If f is linear and $x_1 \neq x_2$, then the slope m is given by

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \tag{3}$$

that is, m is the tangent (trigonometric ratio) of the angle of inclination of f. (See Figure 1-2c).

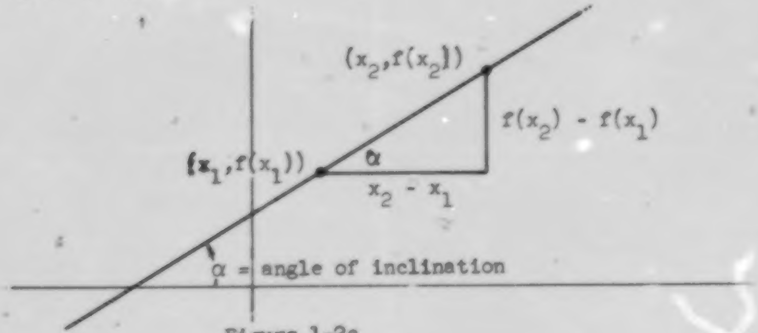


Figure 1-2c

The ratio (3) is, of course, a simple consequence of (2) since

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{m(x_2 - x_1) + b - b}{x_2 - x_1} = m.$$

It will often be convenient to use a slightly different form of the expression (2). This is contained in the following:

(4) The equation of the line through (h,k) with slope m is

$$y = k + m(x - h).$$

Example 1-2a. Find the equation of the line through $(1,2)$ and $(-\frac{1}{2}, \frac{2}{3})$.

The slope of this line is

$$m = \frac{2 - \frac{2}{3}}{1 - (-\frac{1}{2})} = \frac{8}{9};$$

so, using the point $(1,2)$, the form (4) gives the equation

$$y = 2 + \frac{8}{9}(x - 1).$$

Using the point $(-\frac{1}{2}, \frac{2}{3})$ gives the equation

$$y = \frac{2}{3} + \frac{8}{9}(x + \frac{1}{2}).$$

Simple algebra shows that these two equations are just different forms of the equation

$$y = \frac{8}{9}x + \frac{10}{9}.$$

The graph of this line is shown in Figure 1-2d.

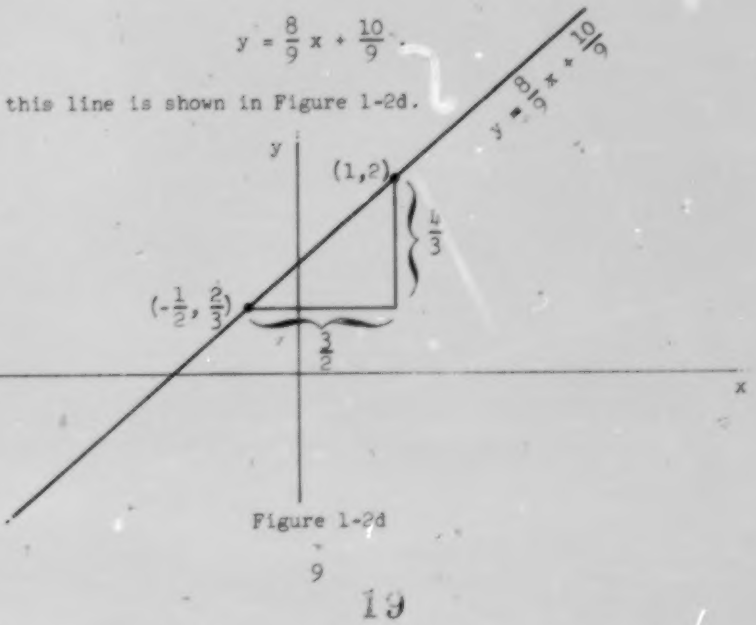


Figure 1-2d

A number of very general concepts have simple and useful formulations for constant and linear functions. Of these we shall now discuss slope as velocity, inverse linear functions, translation and scale change, and proportionality.

Slope as Velocity

The slope of a linear function has various physical interpretations. For example, suppose a man walks north along a long straight road at the uniform rate of 2 miles per hour. At some particular time, say time $t = 0$, this man passed the milepost located one mile north of Baseline Road. An hour before this, which we shall call time $t = -1$, he passed the milepost located one mile south of Baseline Road. An hour after time $t = 0$, at time $t = 1$, he passed the milepost located three miles north of Baseline Road. (See Figure 1-2e.)

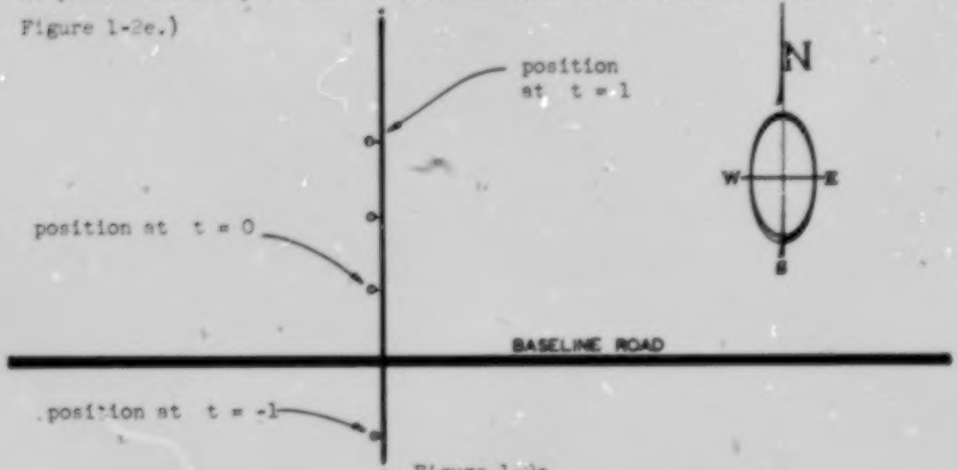


Figure 1-2e

In t hours, the man travels $2t$ miles. Since he is at milepost 1 at time $t = 0$, he must be at milepost $2t + 1$ at time t. Using $f(t)$ to represent the directed distance (in miles) from Baseline Road at time t hours, we see that

$$f : t \to f(t) = 2t + 1$$

describes the man's motion. The graph of f shown in Figure 1-2f thus plots the man's position versus time.

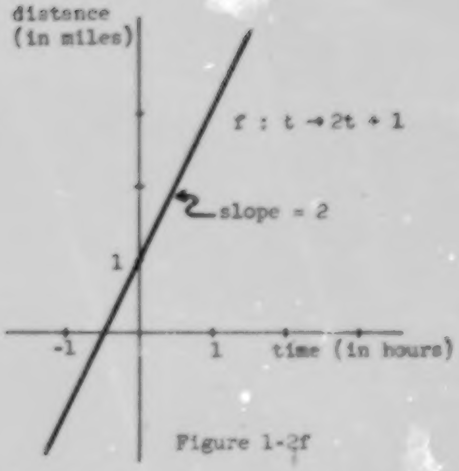


Figure 1-2f

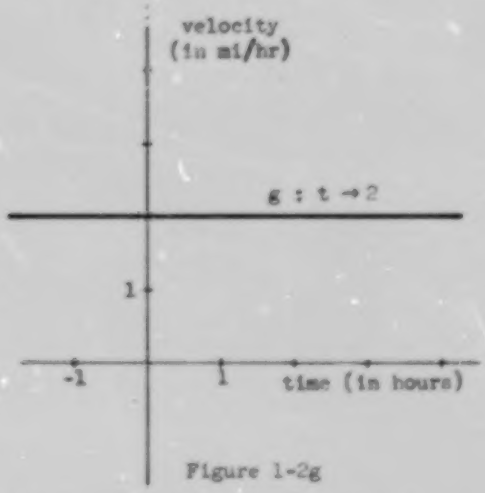


Figure 1-2g

In Figure 1-2g we plot the man's velocity versus time. For all values of t during the time he is walking his velocity is 2 miles per hour. Hence if $g(t)$ is his speed[*] at time t then

$$g : t \to 2,$$

that is, g is a constant function. In this case, the slope function g of the position function f can be interpreted as velocity. We shall encounter this relationship again.

Inverse Linear Functions

We recall that the rule for converting from Centigrade to Fahrenheit is given by the formula

(5) $$F = \frac{9}{5}C + 32$$

and the rule for converting from Fahrenheit to Centigrade is given by the formula

(6) $$C = \frac{5}{9}(F - 32).$$

---

[*] That speed is the absolute value of velocity is something that we shall emphasize later.

Since each of these formulas is useful, it is important to be able to get, say (6) from (5). In (5) to get from C to F we first multiply by $\frac{9}{5}$ and then add 32. Starting with F we must, therefore, first subtract 32 and then divide by $\frac{9}{5}$. That is, to get C we first find $F - 32$ then $\frac{F - 32}{\frac{9}{5}}$ or $\frac{5}{9}(F - 32)$; whence $C = \frac{5}{9}(F - 32)$. In our next example we shall study two functions suggested by the foregoing formulas.

Example 1-2b. Consider the functions $f : x \to \frac{9}{5}x + 32$ and $g : x \to \frac{5}{9}(x - 32)$ and their graphs, sketched on one set of axes in Figure 1-2h. We observe immediately that the slope of f is $\frac{9}{5}$ and the slope of g is the reciprocal of the slope of f.

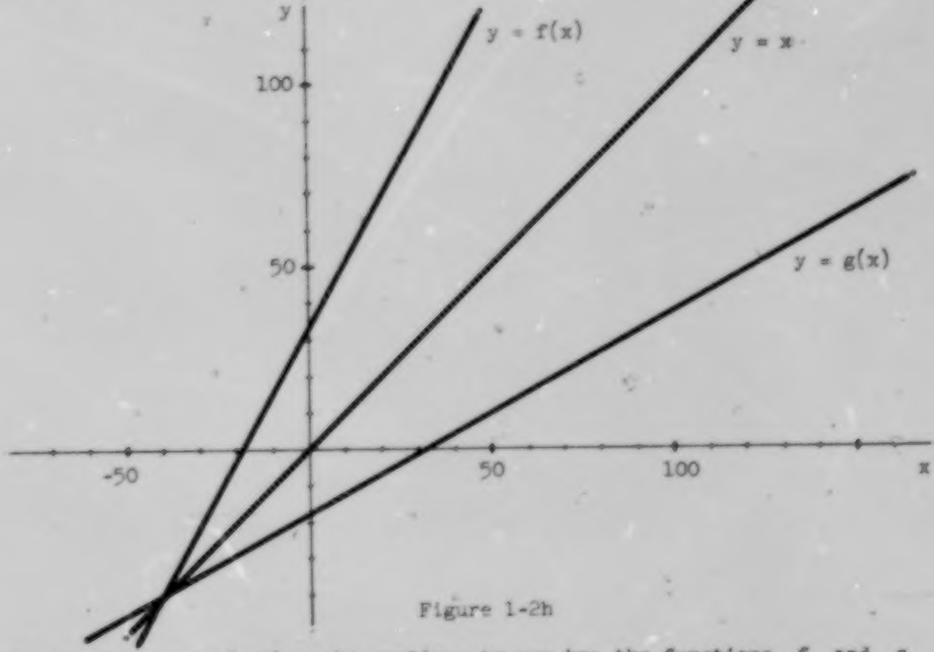


Figure 1-2h

Let us make some further observations to see how the functions f and g and their graphs are related. Consider the equation $y = f(x) = \frac{9}{5}x + 32$. To find the value of y for a given value of x we first multiply x by $\frac{9}{5}$ and then add 32; i.e., on the graph (Figure 1-2i) we first "go up" $\frac{9}{5}x$

to the dotted line and then "go up" 32 more. Finally we "go across" to get $y = \frac{9}{5}x + 32$. To go from y back to x we just reverse the arrows of Figure 1-21. We first subtract 32 to get $\frac{9}{5}x$ and then divide by $\frac{9}{5}$ to get x; i.e.,

$$x = \frac{y - 32}{\frac{9}{5}} = \frac{5}{9}(y - 32).$$

Let x and y exchange roles and compare this form of the equation for f with the function g.

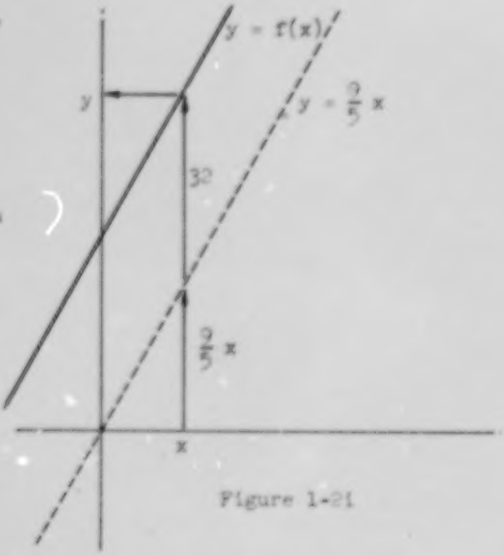


Figure 1-21

The two functions f and g are obviously closely related. In general, if $m \neq 0$, we say that the linear function

$$g : x \to \frac{1}{m}x - \frac{b}{m}$$

is the inverse of the linear function

$$f : x \to mx + b.$$

The slopes of g and f are reciprocals of each other; that is, the product of the slopes is 1

The graph of g can be easily obtained from the graph of f. Suppose that (c,d) lies on the graph of f, so that

$$d = f(c) = mc + b.$$

Solving for c, we have

$$c = \frac{1}{m}d - \frac{b}{m} = g(d).$$

Thus (d,c) lies on the graph of g. The converse of this statement can be similarly established. We summarize in (7).

(7) If g is the inverse of f, then (c,d) lies on the graph of f if and only if (d,c) lies on the graph of g.

Thus the graphs of f and its inverse g are symmetric with respect to the line given by $y = x$. This has a simple geometric interpretation, for it says that the graph of the inverse g can be obtained from the graph of f merely by interchanging the coordinates of each point. This suggests the following way to obtain the graph of the inverse g from that of f. Merely trace the graph of $y = f(x)$ in slow drying ink and then fold carefully along the line given by $y = x$. The wet ink will then trace the graph of the inverse of f. (Consider this mechanical procedure for the graphs of $y = f(x)$ and $y = g(x)$ in Figure 1-2h.)

Translation and Stretch

Let ℓ be the line given by the equation $y = x$, and consider the effect on ℓ of replacing x by $(x - b)$. The new equation is $y = x - b$, which represents the line ℓ' having the same slope as ℓ and intersecting the x-axis at the point $(b,0)$:

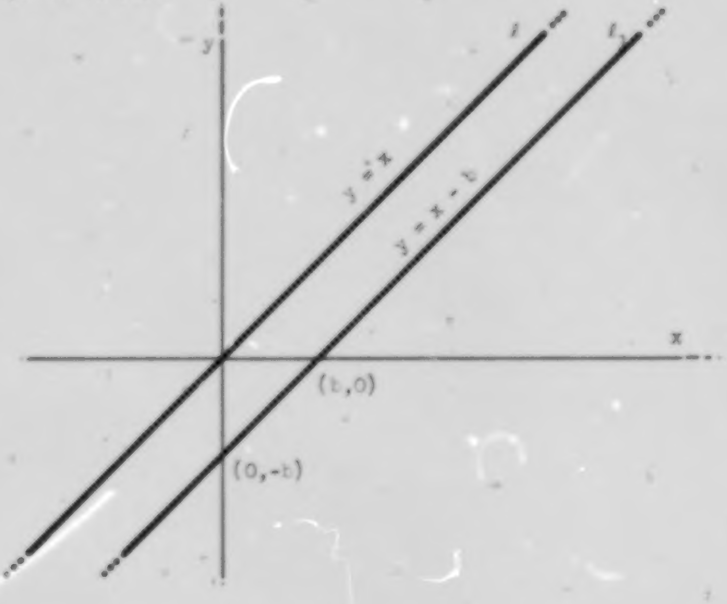


Figure 1-2j

Hence, replacing x by $x - b$ translates or slides ℓ b units to the right without changing its slope (Figure 1-2j).

Now consider the effect on ℓ of replacing x by mx, $m \neq 0$. The new equation is $y = mx$, which represents the line ℓ'' having slope m and passing through the origin.

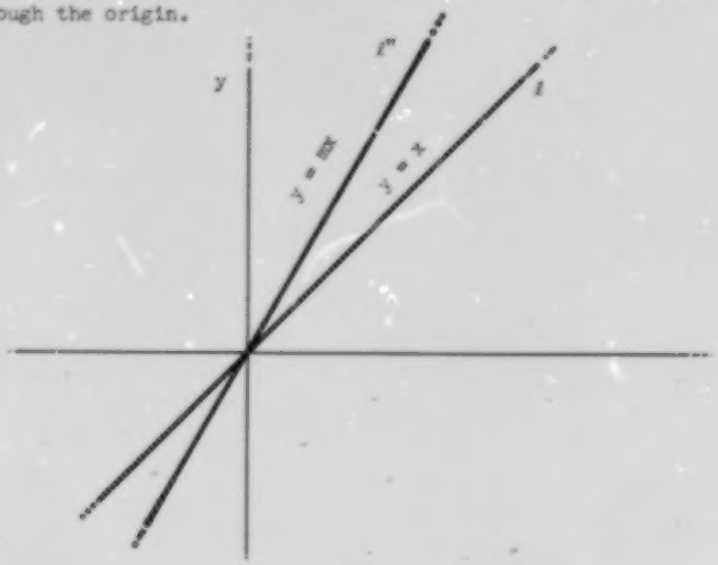


Figure 1-2k

Note that if $m > 1$, replacing x by mx (or equivalently, y by $\frac{y}{m}$) steepens the slope of ℓ, while if $0 < m < 1$ the new line ℓ'' is more horizontal than ℓ. What happens if $m \leq 0$?

Thus, if $m > 1$ the effect of replacing x by mx in the equation $y = x$ is equivalent to stretching the ordinate (y) of each point on ℓ.

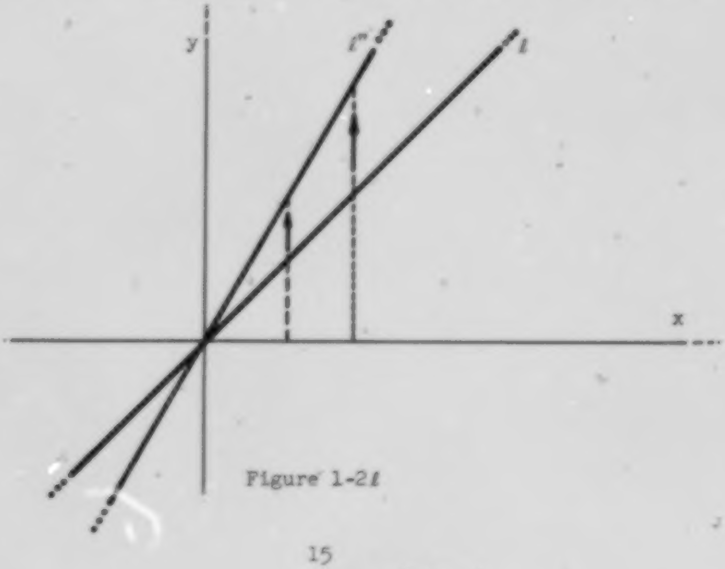


Figure 1-2ℓ

If $0 < m < 1$, each ordinate is "contracted." Note that conceptually, "stretching each ordinate" is no different in this case from "shrinking each abscissa"; the resulting ℓ'' is identical.

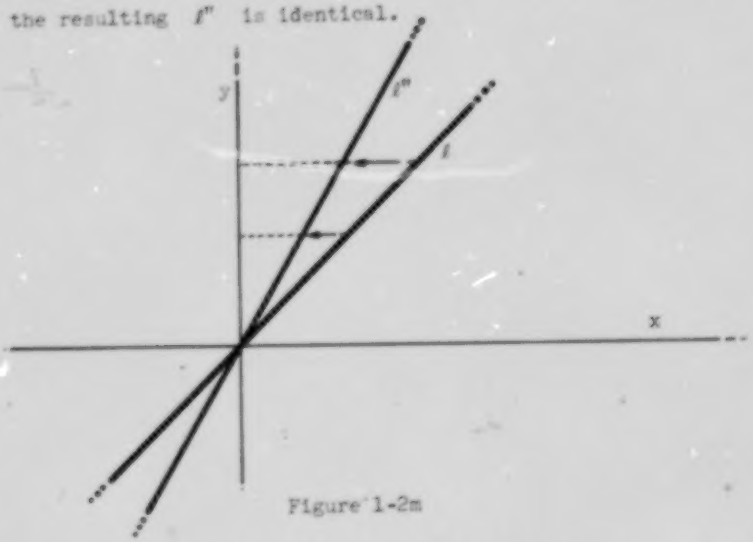


Figure 1-2m

Either way we think about it, the slope of ℓ is changed under such a transformation, and algebraically the stretch can be obtained merely by replacing x by mx (or y by $\frac{y}{m}$) in the equation of ℓ.

The graph of any line given by the equation $y = mx + b$ can be obtained from the graph of $y = x$ by such translation and/or stretching. For example, to obtain the graph of $y = 5x - 4$ from $y = x$, we may first stretch each ordinate of $y = x$ by the factor 5 by replacing x by 5x. The equation becomes $y = 5x$, and its graph is ℓ' shown in Figure 1-2n. Then ℓ' may be translated $\frac{4}{5}$ units to the right by replacing x by $x - \frac{4}{5}$ in the equation of ℓ'.

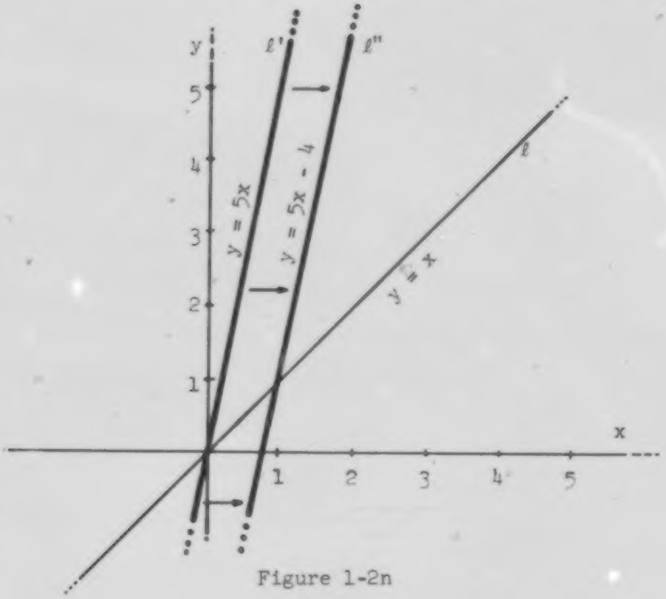


Figure 1-2n

The equation becomes

$$y = 5(x - \frac{4}{5})$$

or

$$y = 5x - 4$$

as desired, and its graph has been obtained from the graph of $y = x$ by translating and stretching. Alternatively, we may first translate the graph 4 units to the right by replacing x by $x - 4$. The equation becomes $y = x - 4$ and the graph is shown in Figure 1-2o. Then we may "shrink" each abscissa by a factor of 5 by replacing x by $5x$. The new equation is $y = 5x - 4$ as before; only our way of thinking about the transformations is different.

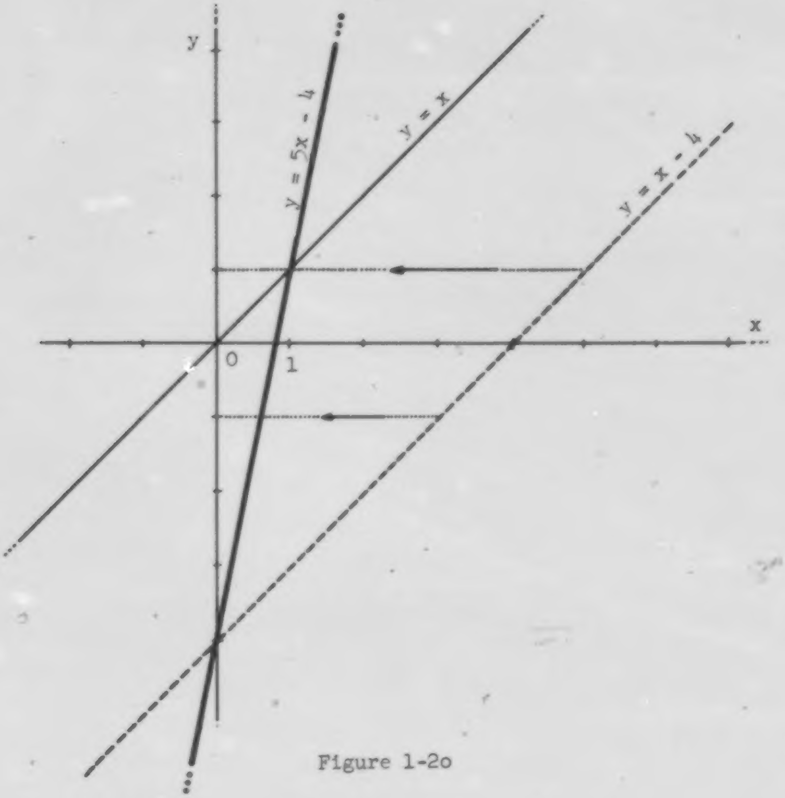


Figure 1-2o

Proportionality

The concept of proportionality is very useful in physics as well as other parts of science. We shall use the idea frequently in this text. To say that y is proportional to x means that there is a number m such that

$$y = mx$$

for all numbers x. The number m is called the constant of proportionality.

Note that if y is proportional to x, then y doubles when x is doubled. The same relationship holds for tripling, halving, etc. In science the experimental observation that y doubles, triples, halves, etc., when x does the same usually leads to the hypothesis that y is proportional to x. Further observation is then used to test such an hypothesis and if no contradictory evidence is found this proportionality is usually stated as a law and thereafter systematically used. For example, if air resistance is neglected it is usually safe to use the assumption that the velocity of a freely falling body is proportional to the time it falls.

Example 1-2c. Assume that the velocity of a free falling body is directly proportional to the time it falls. Suppose that a ball is dropped from the top of a building and attains a velocity of 64 ft./sec. after 2 seconds. How fast will it be falling after 5 seconds (assuming that it hasn't hit the ground by then)?

Since the velocity v ft./sec. is directly proportional to the time t sec, we have $v = mt$, where m is the constant of proportionality. If $v = 64$ when $t = 2$, we get $m = 32$. We obtain the linear function $t \to v = 32t$. When $t = 5$, $v = 160$; therefore, after 5 seconds the velocity is 160 ft./sec.

Exercises 1-2

1. Refer to Figure 1-2f.

   (a) What is the slope of the linear function $t \rightarrow 2t + 1$?

   (b) What are the units of "rise over run" in the graph of $t \rightarrow 2t + 1$?

   (c) Compare the number and units of parts (a) and (b) with the constant function and vertical units of Figure 1-2g.

2. (a) On separate sets of axes sketch graphs of the functions $g : t \rightarrow 32t$ and $g' : t \rightarrow 32$. Indicate vertical and horizontal units appropriate to Example 1-2c.

   (b) What is the slope function of the linear function $g : t \rightarrow 32t$?

   (c) What are the units of the slope of your graph of $g : t \rightarrow 32t$?

   (d) Compare the vertical units for your graph of $g' : t \rightarrow 32$ with your answer to part (c).

   (e) What word from physics is commonly associated with the ratio of units you found in response to part (c)?

3. Assume (as in Example 1-2c) that the velocity of a free falling body is directly proportional to the time it falls. Suppose that a penny is dropped from the top of a tower and attains a velocity of 48 feet per second after $1\frac{1}{2}$ seconds. Determine the impact velocity if the penny hits the ground after $4\frac{1}{2}$ seconds.

4. For the function $f : x \rightarrow 2x + 1$, find

   (a) $f(0)$

   (b) $f(1)$

   (c) $f(-1)$

   (d) for $h \neq 0$, $\dfrac{f(x + h) - f(x)}{h}$

   (e) for $x \neq a$, $\dfrac{f(x) - f(a)}{x - a}$

5. Find the slope of the graph of the function f if, for all real numbers x,

   (a) $f(x) = 3x - 7$

   (b) $f(x) = 6 - 2x$

   (c) $2f(x) = 3 - x$

   (d) $3f(x) = 4x - 2$

6. Find a linear function f whose graph has slope -2 and such that
   (a) $f(1) = 4$
   (b) $f(0) = -7$
   (c) $f(3) = 1$
   (d) $f(8) = -3$

7. Find the slope of the graph of the linear function f if $f(1) = -3$ and
   (a) $f(0) = 4$
   (b) $f(2) = 3$
   (c) $f(5) = 5$
   (d) $f(6) = -13$

8. Find a function whose graph is the line joining the points
   (a) $P(1,1)$, $Q(2,4)$
   (b) $P(-7,4)$, $Q(-5,0)$
   (c) $P(1,3)$, $Q(1,8)$
   (d) $P(1,4)$, $Q(-2,4)$

9. Find the linear function g whose graph passes through the point with coordinates $(-2,1)$ and is parallel to the graph of the function $f : x \to 3x - 5$.

10. Given $f : x \to -3x + 4$, find a function whose graph is parallel to the graph of f and passes through the point
    (a) $P(0,5)$
    (b) $P(-2,3)$
    (c) $P(1,5)$
    (d) $P(-3,-4)$

11. If f is a constant function find $f(3)$ if
    (a) $f(1) = 5$
    (b) $f(8) = -3$
    (c) $f(0) = 4$

12. Do the points $P(1,3)$, $Q(3,-1)$, and $S(7,-9)$ all lie on a single line? Prove your assertion.

13. The graph of a linear function f passes through the points $P(100,25)$ and $Q(101,39)$. Find

    (a) $f(100.1)$

    (b) $f(100.3)$

    (c) $f(101.7)$

    (d) $f(99.7)$

14. The graph of a linear function f passes through the points $P(53,25)$ and $Q(54,-19)$. Find

    (a) $f(53.3)$

    (b) $f(53.8)$

    (c) $f(54.4)$

    (d) $f(52.6)$

15. Find a linear function with graph parallel to the line with equation $x - 3y + 4 = 0$ and passing through the point of intersection of the lines with equations $2x + 7y + 1 = 0$ and $x - 2y + 8 = 0$.

16. Given the points $A(1,2)$, $B(5,3)$, $C(7,0)$, and $D(3,-1)$, show that $ABCD$ is a parallelogram.

17. Find the coordinates of the vertex C of the parallelogram $ABCD$ if AC is a diagonal and the other vertices are the points:

    (a) $A(1,-1)$, $B(3,4)$, $D(2,3)$

    (b) $A(0,5)$, $B(1,-7)$, $D(4,1)$

18. If t is a real number, show that the point $P(t + 1, 2t + 1)$ is on the graph of $f : x \rightarrow 2x - 1$.

19. (a) If you graph the set of all ordered pairs of the form $(t - 1, 3t + 1)$ for any real number t you will obtain the graph of a linear function f. Find $f(0)$ and $f(8)$.

    (b) If you graph the set of all ordered pairs of the form $(t - 1, t^2 + 1)$ for real t, you will obtain the graph of a function f. Find $f(0)$ and $f(8)$.

20. At what temperature do Centigrade degrees equal Fahrenheit degrees?

21. If the slope of a linear function f is negative show that if $x_1 < x_2$ then $f(x_1) > f(x_2)$.

22. Consider the linear functions $f : x \to mx + b$ and $g : x \to \mu x + \beta$ such that $m \neq 0$, $\mu \neq 0$ and $p = g(q)$ if $q = f(p)$ for all real numbers p and q. What is the relationship between m and μ?

23. If $f : x \to mx + b$, $m \neq 0$ find g, the inverse of f.

24. What is the equation of the line perpendicular to the line given by $y = mx + b$, $m \neq 0$, at the y-axis.

25. If $f(2x - 1) = 4x^2 - 8x + 3$, find $f(2x)$.

1-3. Quadratic Functions.

As we discuss the behavior of polynomial functions

$$f : x \to a_0 + a_1x + a_2x^2 + \dots + a_nx^n,$$

we shall use some conventional terminology. If $a_n \neq 0$ we say that the degree of f is n. For example

$$f : x \to 11\frac{1}{2}$$

has degree 0, while

$$f : x \to 2 - 3x + x^2 - x^5$$

has degree 5. This convention assigns no degree to the zero function

$$f : x \to 0. \tag{1}$$

The zero function should not be confused with the zero of a function.

We say that a number r is a zero of the function f or a root of the equation $f(x) = 0$ if $f(r) = 0$. For example, 3 is a zero of the function

$$f : x \to 2x^2 - x - 15$$

or a root of the equation

$$2x^2 - x - 15 = 0,$$

since $f(3) = 0$.

If r is a zero of f then $(r,0)$ lies on the graph of f; that is, the graph of f crosses the x-axis at the point where $x = r$.

Polynomial functions of degree 2 are known as quadratic functions. Let us review some of the properties of quadratics. The zeros of

$$f : x \to c + bx + ax^2, \quad a \neq 0,$$

are given by

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

If $b^2 - 4ac < 0$ these roots are complex numbers. Real roots occur if $b^2 - 4ac \geq 0$.

If $b^2 - 4ac = 0$ then $r_1 = r_2$ and we can factor to obtain

(3) $$f : x \to a(x - r_1)^2,$$

where $r_1 = r_2 = -\frac{b}{2a}$. If $b^2 - 4ac > 0$, then $r_1 \neq r_2$ and we can factor to obtain

(4) $$f : x \to a(x - r_1)(x - r_2).$$

In the following three examples we illustrate the graph of f for each of these cases. In the final example we review a method for graphing quadratics by translation and change of scale.

Example 1-3a. Graph $f : x \to 1 + x + x^2$.

In this case $a = b = c = 1$, so

$$b^2 - 4ac < 0$$

and f has no real zeros. We should expect that the graph of f doesn't cross the x-axis. In fact, the graph of f lies entirely above the x-axis. We can show this by "completing the square" to obtain

$$\begin{aligned} 1 + x + x^2 &= \frac{3}{4} + (\frac{1}{4} + x + x^2) \\ &= \frac{3}{4} + (\frac{1}{2} + x)^2. \end{aligned}$$

Since $(\frac{1}{2} + x)^2 > 0$ unless $x = -\frac{1}{2}$, we see that

$$f(x) > \frac{3}{4}, \quad \text{if} \quad x \neq -\frac{1}{2}$$

while

$$f(-\frac{1}{2}) = \frac{3}{4}.$$

Thus the graph of f lies above the line given by $y = \frac{3}{4}$, touching this line at the point $(-\frac{1}{2}, \frac{3}{4})$. Furthermore, as x increases beyond the value $x = -\frac{1}{2}$, the quantity $(\frac{1}{2} + x)^2$ becomes very large as x becomes large. Also as x decreases (to the left of $x = -\frac{1}{2}$) the quantity $(\frac{1}{2} + x)^2$ increases, becoming very large as we assign numerically large negative values to x. Thus, without plotting points other than $(-\frac{1}{2}, \frac{3}{4})$ we can conclude that the graph of f appears as shown in Figure 1-3a. Of course, a more accurate picture can be obtained by plotting some points $(x, f(x))$.

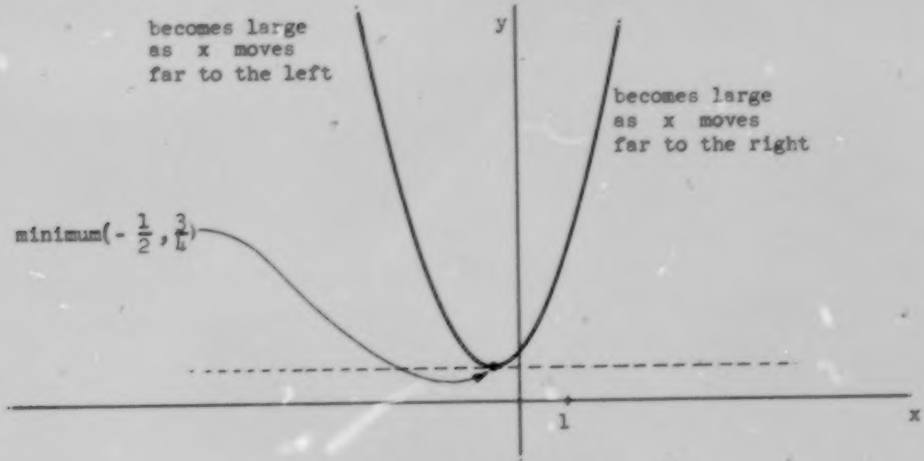


Figure 1-3a. $f : x \to 1 + x + x^2$

Example 1-3b. Graph $f : x \to 4 - 4x + x^2$.

In this case $c = 4$, $b = -4$, $a = 1$ so that

$$b^2 - 4ac = (-4)^2 - 4 \cdot (1)(4) = 0$$

and f has the single zero $x = 2$. Therefore, we can write

$$f : x \to (x - 2)^2.$$

The quantity $(x - 2)^2 > 0$ if $x \neq 2$ so that we have $f(x) > 0$ if $x \neq 2$ and $f(2) = 0$. Therefore, the graph lies above the x-axis, touching this axis at the point $(2,0)$. As x increases to the right of $x = 2$ or as x decreases to the left of $x = 2$, the quantity $(x - 2)^2$ increases, becoming very large as x moves far away from $x = 2$. As in the previous example, this gives us enough information to quickly sketch the graph of f, shown in Figure 1-3b.

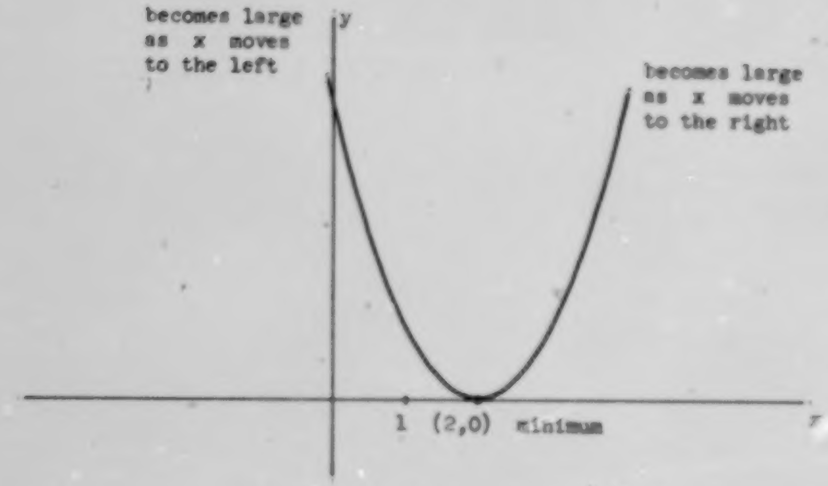


Figure 1-3b. $f : x \rightarrow 4 - 4x + x^2$

Example 1-3c. Graph $f : x \rightarrow 1 - x - 2x^2$.

Since $c = 1$, $b = -1$, $a = -2$, we have

$$b^2 - 4ac = (-1)^2 - 4(-2)(1) = 9.$$

The zeros of f are

$$r_1 = \frac{-(-1) + \sqrt{9}}{-4} = -1 \quad \text{and} \quad r_2 = \frac{-(-1) - \sqrt{9}}{-4} = \frac{1}{2};$$

we can write f in the form

$$f : x \rightarrow -2(x + 1)(x - \tfrac{1}{2}).$$

The graph of f crosses the x-axis at the two points $(-1,0)$ and $(\frac{1}{2}, 0)$. If $x < -1$, each of the quantities $x + 1$ and $x - \frac{1}{2}$ is negative. Upon multiplying by -2 we see that

$$f(x) < 0 \quad \text{if} \quad x < -1.$$

Similarly, we could argue that

$$f(x) > 0 \quad \text{if} \quad -1 < x < \frac{1}{2}$$

and

$$f(x) < 0 \quad \text{if} \quad x > \frac{1}{2}.$$

Further arguments show that as x decreases to the left of -1 or increases to the right of $\frac{1}{2}$, the values $f(x)$ decrease, getting far below the x-axis as x moves far to the right or left.

In the interval $-1 < x < \frac{1}{2}$, the graph of f lies above the x-axis. We can "complete the square" to obtain

$$\begin{aligned} 1 - x - 2x^2 &= -2(x^2 + \tfrac{1}{2}x - \tfrac{1}{2}) \\ &= -2(x^2 + \tfrac{1}{2}x + \tfrac{1}{16} - \tfrac{1}{2} - \tfrac{1}{16}) \\ &= -2(x + \tfrac{1}{4})^2 + \tfrac{9}{8}. \end{aligned}$$

This expression has its greatest value when $x = -\frac{1}{4}$ so that $(-\frac{1}{4}, \frac{9}{8})$ is the highest point on the graph of f. (See Figure 1-3c.)

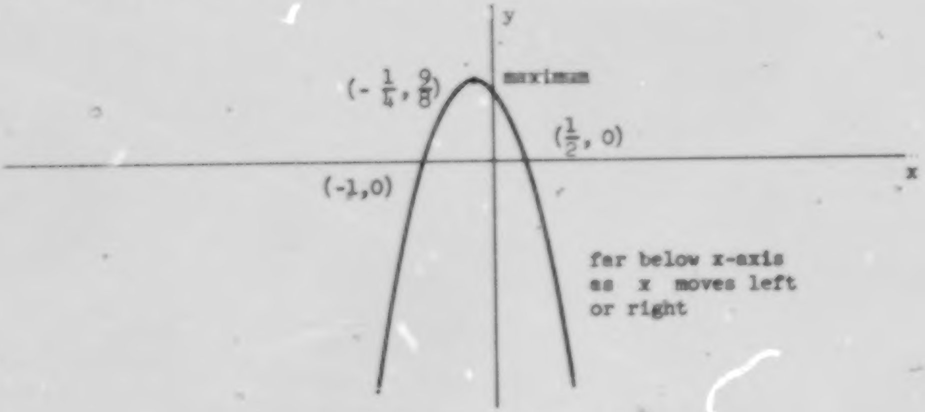


Figure 1-3c. $f : x \rightarrow 1 - x - 2x^2$.

Translation and Stretching of Parabolas

Just as any line with positive slope may be obtained from the graph of $y = x$ by appropriate translations and stretches, any parabola may be obtained from the graph of $y = x^2$ by similar transformations.

For example, to obtain the graph of $y = 14 + 12x + 2x^2$ from the graph of $y = x^2$ we first rewrite the equation as $y = 2(x + 3)^2 - 4$ by completing the square as follows:

$$\begin{aligned} 10 + 12x + 2x^2 &= 2(x^2 + 6x + 7) \\ &= 2(x^2 + 6x + 9 - 2) \\ &= 2(x + 3)^2 - 4. \end{aligned}$$

In this form, the appropriate translations and stretches are readily determined. First, translate the graph of $y = x^2$ three units to the left by replacing x by $(x + 3)$. The new parabola has the equation $y = (x + 3)^2$. (See Figure 1-3d.)

Then "stretch" each ordinate of the graph of $y = (x + 3)^2$ by replacing y by $\frac{y}{2}$. The graph of $y = 2(x + 3)^2$ is shown in Figure 1-3e.

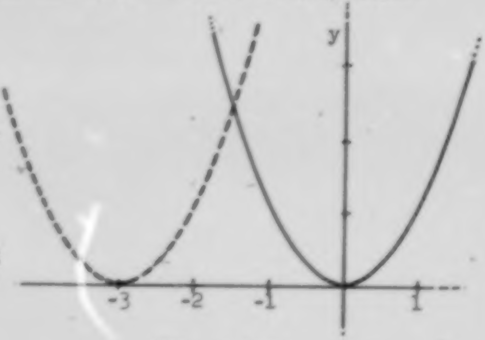


Figure 1-3d

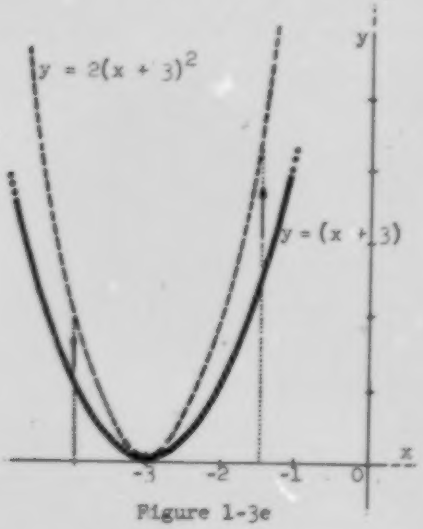


Figure 1-3e

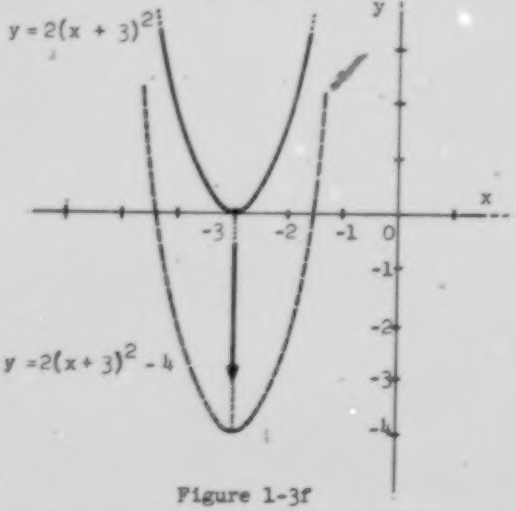
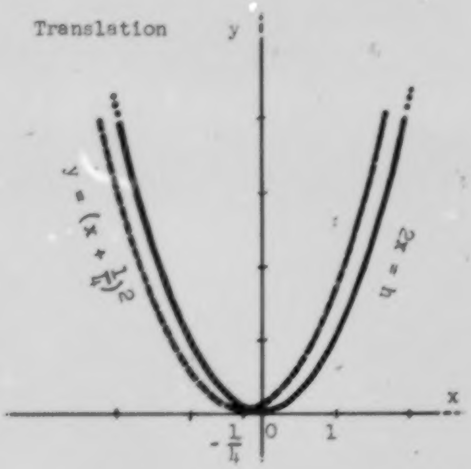


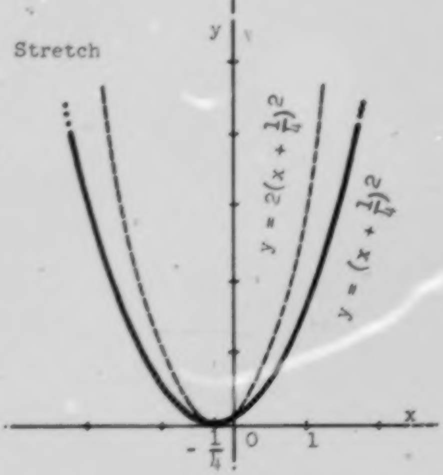
Figure 1-3f

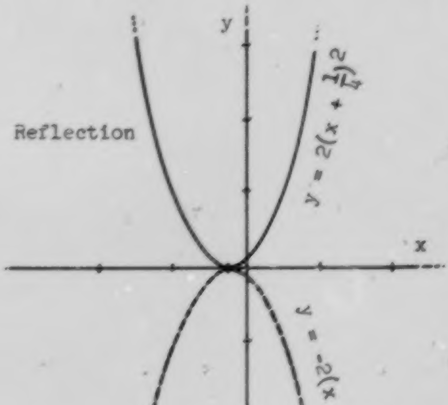
Finally, one more translation, 4 units down is required to obtain the desired graph. This is achieved by replacing y by $(y + 4)$ in the equation $y = 2(x + 3)^2$. See Figure 1-3f for the graph of $y = 2(x + 3)^2 - 4$.

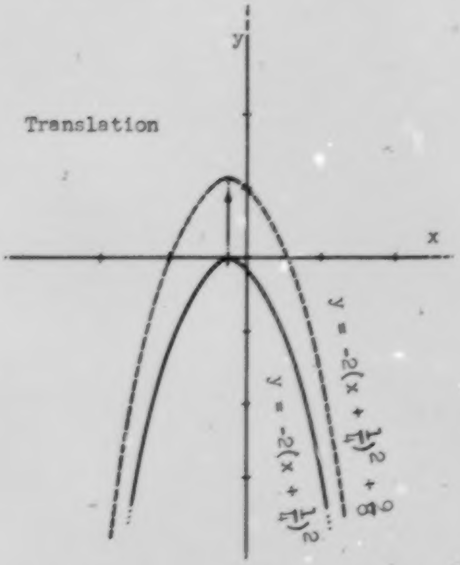
If the coefficient of the x^2 term in the original equation had been negative, as in Example 1-3c, where $y = 1 - x - 2x^2$, one more transformation would have been required to obtain the graph from the graph of $y = x^2$. Completing the square gives $y = -2(x + \frac{1}{4})^2 + \frac{9}{8}$ and in this form we see we must first translate the graph of $y = x^2$, $\frac{1}{4}$ unit to the left by replacing x by $x + \frac{1}{4}$, giving the graph of $y = (x + \frac{1}{4})^2$. Then stretching each ordinate by replacing y by $\frac{y}{2}$ produces the graph of $y = 2(x + \frac{1}{4})^2$. Now the extra transformation, reflecting the graph in the x-axis by replacing y by $-y$, gives the graph of $y = -2(x + \frac{1}{4})^2$. This is physically equivalent to folding the graph along the x-axis or "flipping" the graph about the axis. Finally, translating this graph $\frac{9}{8}$ units up by replacing y by $y - \frac{9}{8}$ gives the desired graph.

In general, the graph of any quadratic function can be obtained from the graph of $y = x^2$ by such a sequence of translations, reflections and stretches. Hence, any quadratic function $f : x \rightarrow ax^2 + bx + c$ represents merely a translated, reflected, and/or stretched image of the standard parabola given by the equation $y = x^2$.


Translation


Stretch


Reflection


Translation

Exercises 1-3

1. Consider the function

$$f : x \rightarrow ax^2 + bx + c.$$

Name the type of function f is, if

(a) $a = 0, \quad b = 0, \quad c \neq 0$

(b) $a = 0, \quad b \neq 0, \quad c = 0$

(c) $a = 0, \quad b \neq 0, \quad c \neq 0$

(d) $a \neq 0, \quad b = 0, \quad c = 0$

(e) $a \neq 0, \quad b \neq 0, \quad c = 0$

(f) $a \neq 0, \quad b \neq 0, \quad c \neq 0$

2. Aristotle claimed that the speed of a free falling object depends on the weight of the object as well as the length of time it falls. Galileo discovered that the speed of a free falling object depends only on how long it falls, and, in particular, that speed v ft./sec. is directly proportional to time t seconds.

(a) A ball is dropped from the top of a building and attains a speed of 64 ft./sec. after 2 seconds. How fast will it be falling after 5 seconds?

(b) A life raft is dropped from a helicopter and hits the water after 10 seconds. If the raft is falling at a speed of 64 ft./sec. after 2 seconds, determine how fast it is going as it hits the water.

3. Galileo discovered that the distance traveled by a falling body of any weight depends only on the length of time in which it has been falling. Specifically it was discovered that the number of feet fallen is directly proportional to the square of the number of seconds elapsed.

(a) Suppose we timed the fall of a ball from the top of a building 400 feet high and discovered that the ball hit the ground after 5 seconds. Find how long it would take for the ball to hit the ground if it were dropped from a building 144 feet high.

(b) Suppose that a ball is dropped from a television tower and hits the ground after 10 seconds. Previously we discovered that the ball hit the ground after 5 seconds when it was dropped from a building 400 feet high. How high is the television tower?

4. For each of the following pairs of functions, (u,v) is on the graph of f and (u,w) is on the graph of g. Determine which is correct: $v > w$, $v = w$, $v < w$.

   (a) $f : x \to 2x^2$

   $g : x \to -2x^2$

   (b) $f : x \to \frac{1}{2}x^2$

   $g : x \to 2x^2$

   (c) $f : x \to -\frac{1}{2}x^2$

   $g : x \to -2x^2$

5. On the same set of axes sketch the graphs of the functions $x \to 5x^2$, x^2, and $\frac{1}{10}x^2$.

6. Describe the location of the points (p,q) and $(-p,q)$ on the graph of $y = ax^2$, relative to each other and the coordinate axes.

7. A ball is dropped from a 47th story window of the Time-Life Building in New York City. Its distance s feet above the Avenue of the Americas after t seconds is described by the equation

$$s = 576 - 16t^2,$$

which serves to define the function

$$f : t \to 576 - 16t^2.$$

   (a) Determine how many feet above the pavement the ball is after falling the first 2 seconds.

   (b) How high above the ground is the ball after zero seconds of falling?

   (c) How high above the Avenue of the Americas is the 47th story window of the Time-Life Building from which we dropped the ball?

   (d) If $f : t \to 576 - 16t^2$, evaluate $f(4)$.

   (e) Four seconds after it is dropped from the 47th story window, how far is the ball from the pavement?

   (f) Find the value of t for which $16t^2 = 576$ and $t > 0$.

   (g) Determine how long it would take for a ball dropped from a 47th story window of the Time-Life Building to hit the pavement below.

8. A ball is dropped from the top of the Fidelity Union Tower in Dallas, Texas. After t seconds the height s ft. of the ball above the ground is given by

$$s = 400 - 16t^2.$$

(a) What is the height of Fidelity Union Tower?

(b) How long does it take for the ball to reach the ground?

9. The Woolworth Building in New York City is about 784 feet high. A ball is dropped from the top of the Woolworth Building so that its distance s feet above the ground after t seconds is described by the equation

$$s = at^2 + c.$$

(a) Relating your experience with other problems of this type to this problem, try to determine appropriate values for a and c.

(b) How long does it take for the ball to reach the ground?

10. The vertex of the parabola given by $y = ax^2 + c$ is the point ________. If > 0 the graph of the function $x \rightarrow ax^2 + c$ opens ________ (upward,downward), and the vertex of the parabola whose equation is $y = ax^2 + c$ is the ________ (highest,lowest) point.

The graph of the equation $y = ax^2 + c$, where a is a non-negative real number and $c > 0$ is always a ________ which is symmetric to the ________, congruent to the graph of $x \rightarrow ax^2$ and c units ________ (above,below) the parabola given by $y = ax^2$.

11. A flowerpot falls from a 75th story windowsill of the Chrysler Building in New York City. We know that after t seconds the height s feet of the flowerpot above the ground is given by the equation

$$s = 1024 - 16t^2.$$

(a) How long does it take for the flowerpot to hit the sidewalk at the corner of Lexington Avenue and Forty Second Street directly beneath the window?

(b) The distance from the 75th story windowsill to the roof of the Chrysler Building is 22 feet. How tall is the Chrysler Building?

12. Suppose that a ball is thrown straight up from the ground with an initial speed of 64 ft./sec. and is not acted upon by the force of gravity. What is its height above the ground after 10 seconds?

A ball is thrown from ground level straight up with an initial speed of 64 ft./sec. Its distance s feet above the ground after t seconds is described by the equation

$$s = 64t - 16t^2,$$

which specifies the function f such that $s = f(t)$.

(a) What is the value of s when $t = 1$?

(b) Evaluate $f(2)$.

(c) What is the height of the ball above the ground after 3 seconds? after 0 seconds?

(d) Sketch the graph of $s = 64t - 16t^2$.

(e) What is the path of the ball?

(f) What is the name of the graph of the function $f : t \rightarrow -16t^2 + 64t$?

13. Suppose that a pellet is projected straight up and after a while comes straight down via the same vertical path to the place on the ground from which it was launched. After t seconds the distance s ft. of the pellet above the ground is described by the equation

$$s = 160t - 16t^2,$$

which defines the function

$$f : t \rightarrow 160t - 16t^2.$$

(See Exercises 1-1, No. 2.) Sketch the graph of f on the interval $0 \leq t \leq 10$.

14. The product of two consecutive integers is zero. What could the integers be?

15. Suppose that you are standing close to the edge on the top of a building 80 feet tall. You throw a ball upward with an initial speed of 64 ft./sec. in a nearly vertical path. After t seconds the height s feet of the ball above the ground is given by the function

$$t \to s = 80 + 64t - 16t^2.$$

(a) How long does it take for the ball to reach the ground?

(b) How high above the building is the ball after one second? after three seconds?

(c) The ball passes the edge of the top of the building from which it was thrown as it falls to the ground. After how many seconds does this occur?

(d) After how many seconds does the ball reach its maximum height?

(e) How high above the building does the ball go?

16. For each of the following pairs of equations, given that (u,v) is on the graph of the first equation and that (u,w) is on the graph of the second determine the values of u for which $v < w$, $v = w$, $v > w$.

(a) $y = 3(x - 4)^2$
$y = -3(x - 4)^2$

(b) $y = 3(x - 4)^2$
$y = 3(x + 4)^2$

17. Compare the graph of $x \to \frac{1}{2}(x - 3)^2 + 2$ with that of $x \to \frac{1}{2}x^2$ on the listed characteristics by completing the following chart.

Function	$x \to \frac{1}{2}x^2$	$x \to \frac{1}{2}(x - 3)^2 + 2$
(a) Name of graph		
(b) Opens (upward or downward)		
(c) Equation of axis		
(d) Coordinates of extremum point (vertex)		
(e) Type of extremum (minimum or maximum)		

18. Sketch the graphs of $x \to \frac{1}{2}x^2$ and $x \to \frac{1}{2}(x - 3)^2 + 2$ on the same set of coordinate axes. Check your answers for Number 17 against your graphs.

19. (a) Using one set of coordinate axes plot the graphs of the following.

    (i) $y = -x^2$

    (ii) $y = -x^2 + 4x + 5$

    (iii) $y = -x^2 - 2x + 3$

    (b) On one set of coordinate axes quickly sketch the graphs of

    (i) $F : x \rightarrow -x^2$

    (ii) $f : x \rightarrow -(x - 2)^2 + 9$

    (iii) $g : x \rightarrow -(x + 1)^2 + 4$

20. Consider the functions $f : x \rightarrow ax^2$ and $g : x \rightarrow a(x - h)^2 + k$. Let (p,q) be a point on the graph of f.

    (a) We know that $f(p) = q$. Another equation relating p and q is $q =$ ________.

    (b) We want to show that the point $(p + h, q + k)$ lies on the graph of g. Show that $g(p + h) = q + k$.

    (c) To every point (p,q) on the graph of f there corresponds the point $(p + h,$ ________$)$ on the graph of g.

    (d) In particular, we see that the vertex (___,___) of the parabola given by $y = ax^2$ corresponds to the vertex (___,___) of the parabola given by $y = a(x - h)^2 + k$.

21. Determine the coordinates of the vertex and the equation of the axis of the parabola given by each of the following equations.

    (a) $y = 2(x - 3)^2 + 4$

    (b) $y = -2(x - 3)^2 + 4$

    (c) $y = (x + 3)^2$

    (d) $y = -\frac{1}{2}(x - 1)^2 - 1$

    (e) $y = 3(x + 1)^2 + 2$

    (f) $y = \frac{1}{5}(x - 2)^2 - 3$

22. Determine the extremum point of each graph in Number 21 and tell whether it is a maximum or minimum.

23. For each of the following pairs of equations, given that (u,v) is on the graph of the first equation and (u,w) is on the graph of the second, determine the values of u for which $v < w$, $v = w$, $v > w$.

(a) $y = 2(x - 3)^2 + 6$

$y = 2(x - 3)^2 - 6$

(b) $y = 2(x - 3)^2 + 6$

$y = -2(x - 3)^2 - 6$

(c) $y = 2(x - 3)^2 + 6$

$y = 2(x + 3)^2 + 6$

24. Write each of the following equations in the form $y = a(x - h)^2 + k$.

(a) $y = x^2 - 6x + 9$

(b) $y = 2(x^2 - 6x + 9)$

(c) $y = 2x^2 - 12x + 18$

(d) $y = 2(x^2 - 6x + 9) + 4$

(e) $y = 2x^2 - 12x + 22$

(f) $y = -2(x^2 - 6x + 9)$

(g) $y = -2(x^2 - 6x + 9) + 4$

(h) $y = -2x^2 + 12x - 14$

(i) $y = x^2 + 6x + 9$

(j) $y = x^2 - 2x + 1$

(k) $y = -\frac{1}{2}x^2 + x - \frac{1}{2}$

(ℓ) $y = -\frac{1}{2}x^2 + x - \frac{3}{2}$

(m) $y = x^2 + 2x + 1$

(n) $y = 3x^2 + 6x + 9$

(o) $y = 3x^2 + 6x + 11$

(p) $y = x^2 - 4x + 4$

(q) $y = \frac{1}{5}(x^2 - 4x + 4)$

(r) $y = \frac{x^2}{5} - \frac{4}{5}x - \frac{11}{5}$

25. For the function $f : x \to ax^2 + bx + c$, $a \neq 0$, prove that if $f(x_1) = f(x_2) = 0$, then

$$x_1 \text{ or } x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1-4. Polynomial Functions of Degree Higher Than One

Most of our initial work with polynomial functions will be concerned with two related problems:

> Problem 1: Given a function f and any number x in its domain, find $f(x)$.
>
> Problem 2. Given a function f and any number y in its range, find all numbers x such that $f(x) = y$; in particular, find those values of x for which $y = 0$, i.e., the zeros of f.

Later we shall study the second of these two problems.

Soon we shall also develop techniques for determining maximum and minimum points, intervals of increase and decrease and behavior for large values of x for polynomial functions of degree higher than two. As in the quadratic examples of the previous section these techniques will enable us to sketch graphs of higher degree polynomial functions very quickly. For now we consider Problem 1. To graph polynomial functions and find the solutions of polynomial equations, it is important to evaluate a given $f(x)$ for different values of x. For example, to graph

$$f : x \rightarrow 3x^3 - 2x^2 + x - 6,$$

we may want the values $f(x)$ at $x = 0, 1, 2, 3$, etc. Of course, we may obtain these values by direct substitution, doing all of the indicated multiplications and additions. For most values this is tedious. Fortunately, there is an easier way which we shall call synthetic substitution. To understand the method, we shall analyze a few easy examples.

Example 1-4a. Find the value of

$$f(x) = 2x^2 - x + 3 \quad \text{at} \quad x = 4.$$

We write

$$f(x) = (2x - 1)x + 3.$$

When $x = 4$, this becomes

$$[2(4) - 1]4 + 3 = 31.$$

Note that to evaluate our expression, we can

(i) Multiply 2 (the coefficient of x^2) by 4 and add this product to -1 (the coefficient of x);

(ii) Multiply the result of (i) by 4 and add this product to 3 (the constant term).

Example 1-4b. Find $f(3)$, given

$$f(x) = 2x^3 - 3x^2 + 2x + 5.$$

We may write $f(x)$ as

$$(2x^2 - 3x + 2)x + 5$$

or

$$[(2x - 3)x + 2]x + 5.$$

To find the value of this expression when $x = 3$, we may start with the inside parentheses and

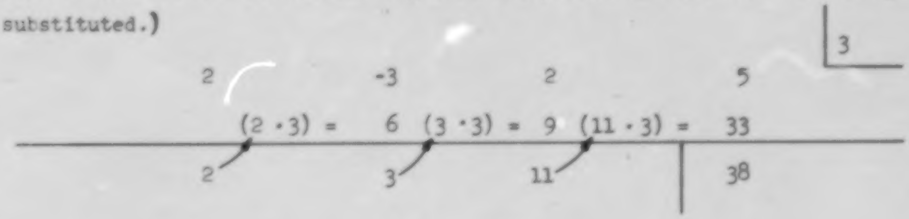
(i) Multiply 2 (the coefficient of x^3) by 3 and add this product to -3 (the coefficient of x^2);

(ii) Multiply the result of (i) by 3 and add this product to 2 (the coefficient of x);

(iii) Multiply the result of (ii) by 3 and add this product to 5 (the constant term).

The result is $f(3) = 38$.

These steps can be represented conveniently by a table whose first row consists of the coefficients of the successive powers of x in descending order. (The number at the far right is the particular value of x being substituted.)



When this tabular arrangement is used, we proceed from left to right. We start the process by rewriting the first coefficient 2, in the third row. Each entry in the second row is 3 times the entry in the third row of the preceding column. Each entry in the third row is the sum of the two entries above it. We note that the result 38, can be checked by direct substitution.

Now let us consider the general cubic polynomial

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0, \quad a_3 \neq 0.$$

When $x = c$, we have

$$f(c) = a_3c^3 + a_2c^2 + a_1c + a_0,$$

which may be written

$$f(c) = [(a_3c + a_2)c + a_1]c + a_0.$$

Again the steps employed in the procedure can be represented in tabular form:

$$\begin{array}{ccccl} a_3 & a_2 & a_1 & a_0 & \underline{|\,c} \\ & a_3c & (a_3c + a_2)c & [(a_3c + a_2)c + a_1]c & \\ \hline a_3 & a_3c + a_2 & (a_3c + a_2)c + a_1 & \multicolumn{1}{|l}{f(c)} & \end{array}$$

As in earlier examples, the number being substituted is written to the right of the entire array.

Example 1-4c. Given $f(x) = 3x^3 - 2x^2 + x - 6$, determine $f(2)$.

$$\begin{array}{cccc|cl} 3 & -2 & 1 & \multicolumn{1}{c}{-6} & & \underline{|\,2} \\ & 6 & 8 & \multicolumn{1}{c}{18} & & \\ \hline 3 & 4 & 9 & 12 & & \end{array}$$

Now 12 is the result sought, namely $f(2)$. This may be checked by direct substitution:

$$f(2) = 3(2)^3 - 2(2)^2 + 2 - 6 = 24 - 8 + 2 - 6 = 12.$$

Example 1-4d. Given $f(x) = x^4 - 3x^2 + 2x - 5$, determine $f(3)$. Note that $a_3 = 0$ and that this number must be written in its appropriate place as one of the detached coefficients in the first row.

$$\begin{array}{ccccc|l} 1 & 0 & -3 & 2 & \multicolumn{1}{c}{-5} & \underline{|\,3} \\ & 3 & 9 & 18 & \multicolumn{1}{c}{60} & \\ \hline 1 & 3 & 6 & 20 & \multicolumn{1}{|c}{55} & \end{array}$$

Thus, $f(3) = 55$, which, as before, may be checked by direct substitution.

With a little care and practice, the second line in the above work can often be omitted when c is a small integer.

Example 1-4e. Given $f(x) = x^4 - x^3 - 16x^2 + 4x + 48$, evaluate $f(x)$ for $x = -3, -2, -1, 0, 1, 2, 3, 4, 5$.

We detach the coefficients. In order to avoid confusion, it is sometimes convenient to write them down at the bottom of a sheet of scratch paper and slide this down, covering at each step the work previously done. As suggested above, we omit the second line in each evaluation and write the value of x we are using adjacent to the answer. The results appear in Table 1-4a.

The last two columns now become a table of $f(x)$ and x. Note that the row that corresponds to $x = 0$ has the same entries as the coefficient row.

Table 1-4a

Coefficients of $f(x) = x^4 - x^3 - 16x^2 + 4x + 48$

1	-1	-16	4	48	
1	-4	-4	16	0	-3
1	-3	-10	24	0	-2
1	-2	-14	18	30	-1
1	-1	-16	4	48	0
1	0	-16	-12	36	1
1	1	-14	-24	0	2
1	2	-10	-26	-30	3
1	3	-4	-12	0	4
1	4	4	24	168	5
				$f(x)$	x

The method described and illustrated above is often called synthetic substitution or synthetic division in algebra books. The word "synthetic" literally means "put together," so you can see how it is that "synthetic substitution" is appropriate here; later we shall illustrate why the process is also called "division." The method gives a quick and efficient means of evaluating $f(x)$, and we are now able to plot the graphs of polynomials more easily than would be the case if the values of $f(x)$ had to be computed by direct substitution.

Example 1-4f. Plot the graph of the polynomial function

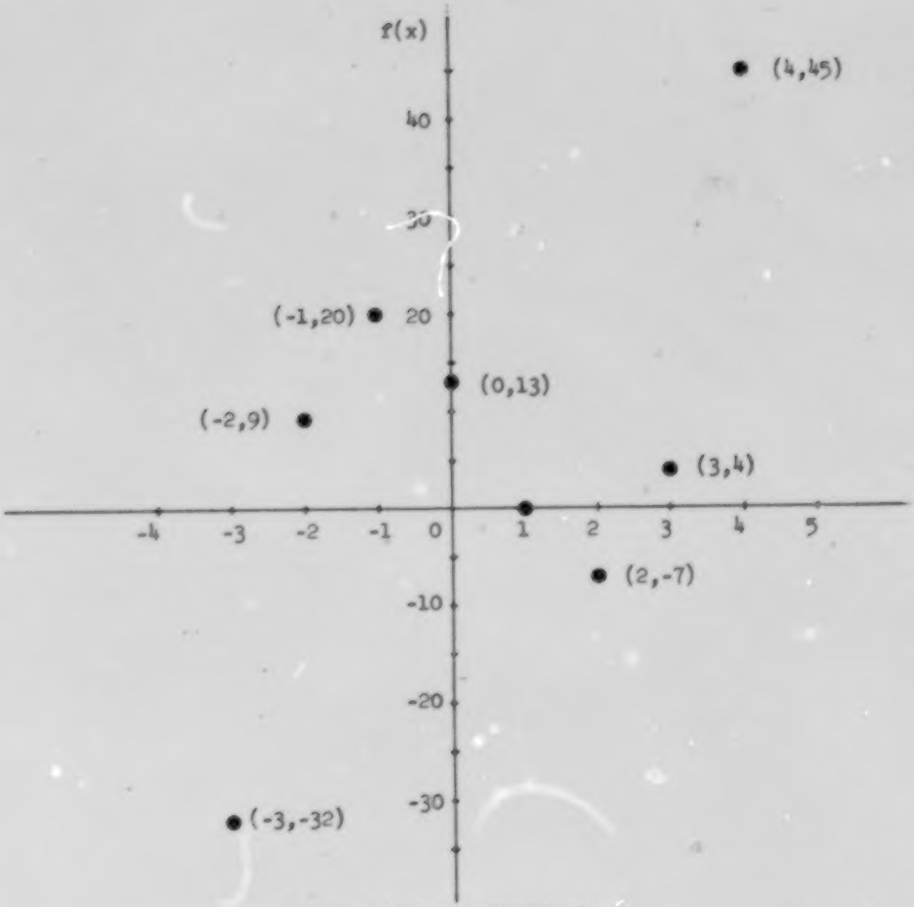
$$f : x \rightarrow 2x^3 - 3x^2 - 12x + 13.$$

We prepare a table of values of x and $f(x)$ by synthetic substitution, and then plot the points whose coordinates $(x,f(x))$ appear in Table 1-4b.

Table 1-4b

Coefficients of $f(x) = 2x^3 - 3x^2 - 12x + 13$				
2	-3	-12	13	
2	-9	15	-32	-3
2	-7	2	9	-2
2	-5	-7	20	-1
2	-3	-12	13	0
2	-1	-13	0	1
2	1	-10	-7	2
2	3	-3	4	3
2	5	8	45	4
			$f(x)$	x

From the table we observe that the points $(x,f(x))$ to be plotted are $(-3,-32)$, $(-2,9)$, $(-1,20)$, etc. These points are located on a rectangular coordinate system as shown in Figure 1-4a. Note that we have chosen different scales on the axes for convenience in plotting.



Some Points on Graph of

$$f : x \rightarrow 2x^3 - 3x^2 - 12x + 13$$

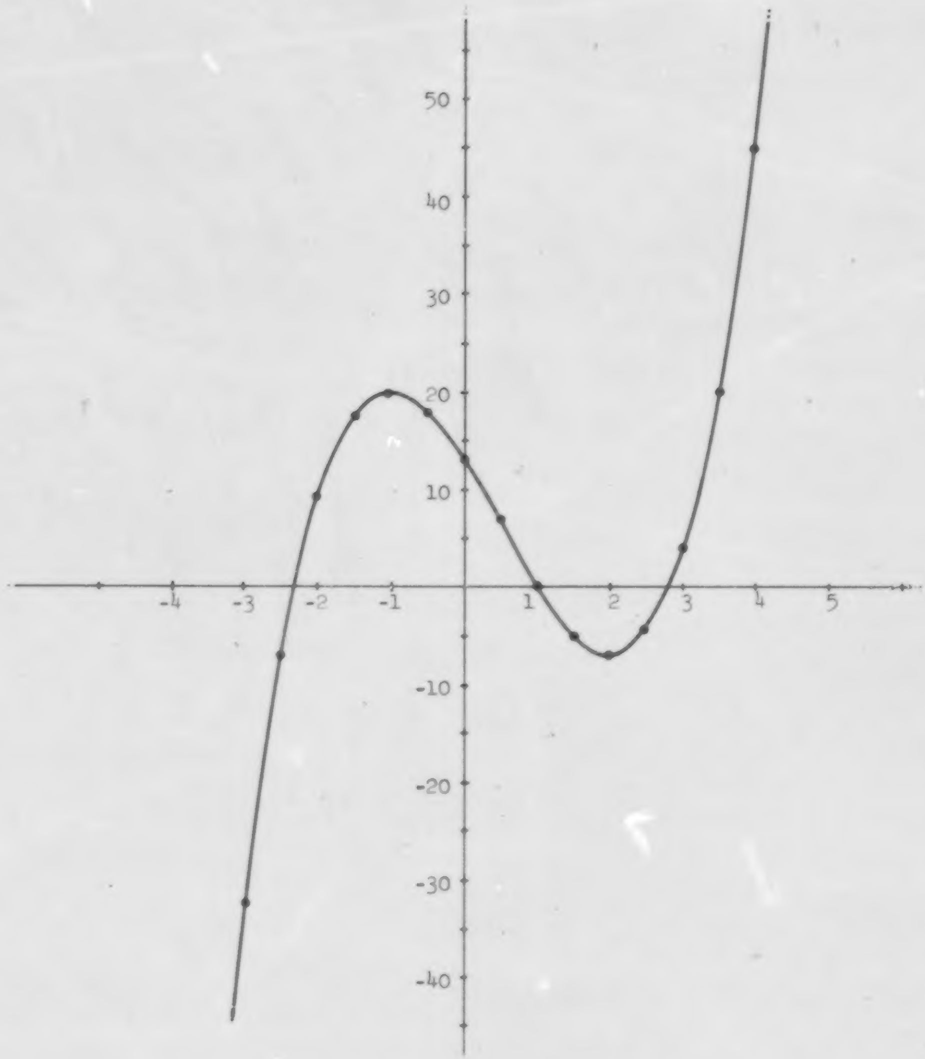
Figure 1-4a

Now the problem is how best to draw the graph. We shall assume that the graph has no breaks. The question remains whether the points we have already plotted are sufficient to give us a fairly accurate picture of the graph, or whether there may be hidden "peaks" and "valleys" not shown thus far. We are not in a position to answer this question categorically at present, but we can shed further light on it by plotting more points between those already located. By use of fractional values of x and the method of synthetic substitution, Table 1-4b is extended as shown in Table 1-4c.

Table 1-4c

Coefficients of $f(x) = 2x^3 - 3x^2 - 12x + 13$				
2	-3	-12	13	
2	-8	8	-7	$-\frac{5}{2}$
2	-6	-3	$\frac{35}{2}$	$-\frac{3}{2}$
2	-4	-10	18	$-\frac{1}{2}$
2	-2	-13	$\frac{13}{2}$	$\frac{1}{2}$
2	0	-12	-5	$\frac{3}{2}$
2	2	-7	$-\frac{9}{2}$	$\frac{5}{2}$
2	4	2	20	$\frac{7}{2}$
			$f(x)$	x

When we fill in these points on the graph, it appears that if we connect the points by a smooth curve, we ought to have a reasonably accurate picture of the graph of f in the interval from -3 to 4. This is shown in Figure 1-4b.



Graph of $f : x \rightarrow 2x^3 - 3x^2 - 12x + 13$

Figure 1-4b

Exercises 1-4

1. Evaluate the following polynomials for the given values of x.

   (a) $f(x) = x^4 + x - 3;\ x = -2, 1, 3$

   (b) $f(x) = x^2 - 3x^3 + x - 2;\ x = -1, -3, 0, 2, 4$

   (c) $g(x) = 3x^3 - 2x^2 + 1;\ x = \frac{1}{2}, \frac{1}{3}, 2$

   (d) $r(x) = 6x^3 - 5x^2 - 17x + 6;\ x = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, 2$

   (e) $s(x) = 6x^3 - 29x^2 + 37x - 12;\ x = 0, 1, 2, 3, 4$

   (f) $F(x) = 3x^4 - 97x^3 + 35x^2 + 8x + 2;\ x = \frac{1}{3}$

   (g) $G(x) = x^{10} - 4x^3 + 10;\ x = 2$

2. If $f(x) = 2x^3 - kx^2 + 3x - 2k$, for what value of k will $f(2) = 4$?

3. If $f(x) = x^3 + kx^2 - 3x$, for what value of k will $f(2) = 2$?

4. (a) Find the value of each of the following functions when $x = -1$.

   (i) $x \to 2$

   (ii) $x \to 2x - 7$

   (iii) $x \to 2x^2 - 5x - 7$

   (iv) $x \to 2x^3 - 3x^2 - 12x + 13$

   (b) Plot the graph of the function

   $$x \to 2x^3 - 9x^2 + 20.$$

   (c) Compare your sketch of part (b) with the graph of $f : x \to 2x^3 - 3x^2 - 12x + 13$ in Figure 1-4b.

5. (a) Find the value of each of the following functions when $x = 1$.

   (i) $x \to 2$

   (ii) $x \to 2x + 1$

   (iii) $x \to 2x^2 - x - 13$

   (iv) $x \to 2x^3 - 3x^2 - 12x + 13$

   (b) Plot the graph of the function

   $$x \to 2x^3 + 3x^2 - 12x.$$

   (c) Compare your sketch of part (b) with the graph of $f : x \to 2x^3 - 3x^2 - 12x + 13$ in Figure 1-4b.

6. Plot the graph of each of the following functions and compare each graph with the graph of $f : x \rightarrow 2x^3 - 3x^2 - 12x + 13$ in Figure 1-4b.

   (a) $x \rightarrow -2x^3 + 3x^2 + 12x - 13$

   (b) $x \rightarrow 2x^3 - 3x^2 - 12x$

7. Plot the graph of each of the following functions.

   (a) $x \rightarrow x^3$

   (b) $x \rightarrow x^3 + 4$

   (c) $x \rightarrow (x - 2)^3$

8. Plot the graph of each of the following functions.

   (a) $x \rightarrow x^3 + 3x^2 + 4$

   (b) $x \rightarrow x^3 + 3x^2$

   (c) $x \rightarrow (x + 1)^3 + 3(x + 1)^2$

9. Sketch the graph of each of the following functions.

   (a) $x \rightarrow 2x^3 - 12x + 18$

   (b) $x \rightarrow 2x^3 - 12x$

   (c) $x \rightarrow 2(x - 3)^3 - 12(x - 3)$

10. Sketch the graph of $x \rightarrow x^4$ over the interval $-\frac{3}{2} \leq x \leq \frac{3}{2}$.

11. Plot the graph of each of the following functions.

    (a) $x \rightarrow x^4 - 2x^3 - 5x^2 + 6x$

    (b) $x \rightarrow -x^4 + 2x^3 + 5x^2 - 6x$

    (c) $x \rightarrow x^4 + 2x^3 - 5x^2 - 6x$

12. For $f : x \rightarrow 44 + 4x - 13x^2 + 18x^3 - 9x^4$ we give some of the functional values in the following table.

x	-2	-1	0	1	2	3
f(x)	-304	0	44	44	0	-304

    (a) From the table estimate the value of x for which the function has a maximum value.

    (b) Sketch the graph of f, with special care given to the interval $0 \leq x \leq 1$.

13. Approximate the maximum value of the function

$$f : x \rightarrow 39 - 640x^2 - 128x^3 - 640x^4.$$

1-5. Remainder and Factor Theorems

Momentarily we shall turn away from graphing and take another look at the process we described in Section 1-4, in order to develop some theorems that will be useful in finding the zeros of polynomial functions. The synthetic substitution used to determine $f(2)$, given

$$f : x \to x^3 - 7x^2 + 3x - 2$$

will be the basis for this development, so let us examine it closely.

$$\begin{array}{cccc|l} 1 & -7 & 3 & -2 & \underline{2} \\ & 2 & -10 & -14 & \\ \hline 1 & -5 & -7 & -16 & \end{array}$$

We rewrite the first row in the synthetic substitution as the given polynomial (by restoring the powers of x), and then attach the same power of x to each entry in a given column. Thus we obtain

$$\begin{array}{cccc} 1x^3 & -7x^2 & +3x & -2 \\ & 2x^2 & -10x & -14 \\ \hline 1x^3 & -5x^2 & -7x & -16 \end{array}$$

The polynomial in the third row is the sum of the two preceding polynomials. Since $f(x) = x^3 - 7x^2 + 3x - 2$ and $f(2) = -16$, the above addition can be written

$$f(x) + 2x^2 - 10x - 14 = x^3 - 5x^2 - 7x + f(2).$$

By factoring, we may write

$$f(x) + 2(x^2 - 5x - 7) = x(x^2 - 5x - 7) + f(2).$$

Solving for $f(x)$, we have

$$f(x) = x(x^2 - 5x - 7) - 2(x^2 - 5x - 7) + f(2),$$

or

$$f(x) = (x - 2)(x^2 - 5x - 7) + f(2).$$

The form of this expression may look familiar. It is, in fact, an example of the division algorithm:

Dividend = (Divisor)(Quotient) + Remainder.

In our example, if $(x - 2)$ is the divisor, then

$$q(x) = x^2 - 5x - 7$$

is the quotient, and $f(2)$ is the remainder. This result may be generalized. It is of sufficient importance to be stated as a theorem.

REMAINDER THEOREM. If $f(x)$ is a polynomial of degree $n > 0$ and if c is a number, then the remainder in the division of $f(x)$ by $x - c$ is $f(c)$. That is,

$$f(x) = (x - c)q(x) + f(c),$$

where the quotient $q(x)$ is a polynomial of degree $n - 1$.

Proof. We shall prove the theorem only in the case of the general cubic polynomial,

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0.$$

Following the pattern of the previous example, to determine $f(c)$ we write the synthetic substitution

$$\begin{array}{cccc|l} a_3 & a_2 & a_1 & a_0 & \underline{c} \\ & a_3c & (a_3c + a_2)c & (a_3c^2 + a_2c + a_1)c & \\ \hline (a_3) & (a_3c + a_2) & (a_3c^2 + a_2c + a_1) & (a_3c^3 + a_2c^2 + a_1c + a_0) & \end{array}$$

As before, writing in the appropriate powers of x, we get

$$\begin{array}{ccccccc} a_3x^3 & + & a_2x^2 & + & a_1x & + & a_0 \\ & + & a_3cx^2 & + & (a_3c + a_2)cx & + & (a_3c^2 + a_2c + a_1)c \\ \hline a_3x^3 & + & (a_3c + a_2)x^2 & + & (a_3c^2 + a_2c + a_1)x & + & (a_3c^3 + a_2c^2 + a_1c + a_0) \end{array}$$

We note that the polynomial in the third row is the sum of the two preceding polynomials, that the polynomial in the first row is $f(x)$ and that $(a_3c^3 + a_2c^2 + a_1c + a_0)$ is $f(c)$. Hence we may write

$$f(x) + c[a_3x^2 + (a_3c + a_2)x + (a_3c^2 + a_2c + a_1)] =$$
$$x[a_3x^2 + (a_3c + a_2)x + (a_3c^2 + a_2c + a_1)] + f(c).$$

Thus we have

$$f(x) = (x - c)[a_3x^2 + (a_3c + a_2)x + (a_3c^2 + a_2c + a_1)] + f(c)$$

or

$$f(x) = (x - c)q(x) + f(c).$$

The process is the same for higher degree polynomials. It gives

$$f(x) = (x - c)q(x) + f(c),$$

where $q(x)$ is a polynomial of degree $n - 1$.

If the remainder $f(c)$ is zero, then the divisor $x - c$ and the quotient $q(x)$ are factors of $f(x)$. Hence, we have as an immediate consequence the Factor Theorem.

FACTOR THEOREM. If c is a zero of a polynomial function f of degree $n > 0$, then $x - c$ is a factor of $f(x)$, and conversely.

We know from the Remainder Theorem (applied to a cubic function) that there exists a polynomial $q(x)$ of degree $n - 1$ such that

$$f(x) = (x - c)q(x) + f(c).$$

If c is a zero of f, then $f(c) = 0$ and

$$f(x) = (x - c)q(x).$$

Hence, $x - c$ is a factor of $f(x)$, by definition.

Conversely, if $x - c$ is a factor of $f(x)$, then by definition there is a polynomial $q(x)$ such that

$$f(x) = (x - c)q(x).$$

For $x = c$, we obtain

$$f(c) = (c - c)q(c) = 0,$$

and hence c is a zero of f.

Example 1-5a. Find the quotient and remainder if

$$f(x) = 2x^3 - 6x^2 + x - 5$$

is divided by $x - 3$.

$$\begin{array}{rrrrl} 2 & -6 & 1 & -5 & \quad \lfloor 3 \\ & 6 & 0 & 3 & \\ \hline 2 & 0 & 1 & -2 & \end{array}$$

Hence,

$$q(x) = 2x^2 + 1,$$

$$f(3) = -2,$$

and

$$2x^3 - 6x^2 + x - 5 = (x - 3)(2x^2 + 1) - 2.$$

Example 1-5b. Show that $x - 6$ is a factor of

$$f(x) = x^3 - 6x^2 + x - 6,$$

and find the associated $q(x)$.

$$\begin{array}{rrrrl} 1 & -6 & 1 & -6 & \quad \lfloor\underline{\;6\;} \\ & 6 & 0 & 6 & \\ \hline 1 & 0 & 1 & 0 & \end{array}$$

Here, $f(6) = 0$, $q(x) = x^2 + 1$, and

$$f(x) = (x - 6)(x^2 + 1).$$

In testing for the divisibility of a polynomial by $ax + b$, $a \neq 0$, we write

$$ax + b = a(x + \frac{b}{a}) = a[x - (-\frac{b}{a})]$$

and see whether $f(-\frac{b}{a}) = 0$. By the Factor Theorem, $ax + b$ is a factor of $f(x)$ if and only if $f(-\frac{b}{a}) = 0$. (Note that $-\frac{b}{a}$ is the root of $ax + b = 0$.)

In applying the Factor Theorem, it may sometimes be easier to compute $f(c)$ by direct substitution, rather than by the method of synthetic substitution. Thus, to show that $x - 1$ is a factor of

$$f(x) = 2x^{73} - x^{37} - 1,$$

we note that $$f(1) = 2 - 1 - 1 = 0.$$

Evaluating $f(1)$ by the synthetic substitution method would take considerably longer.

At this point you may wonder what to do when confronted with a polynomial such as

$$8x^4 - 28x^3 - 62x^2 + 7x + 15,$$

which you might like to factor. Note that the Factor Theorem is only a testing device. It does not locate zeros of polynomial functions. (Methods, other than blind guessing, for doing this will be developed later.)

Exercises 1-5

1. Find $q(x)$ and $f(c)$ so that $f(x) = (x - c)q(x) + f(c)$ if,
   - (a) $f(x) = 3x^3 + 4x^2 - 10x - 15$ and $c = 2$
   - (b) $f(x) = x^3 + 3x^2 + 2x + 12$ and $c = -3$
   - (c) $f(x) = -2x^4 + 3x^3 + 6x - 10$ and $c = 3$
   - (d) $f(x) = 2x^3 - 3x^2 + 5x - 2$ and $c = \frac{1}{2}$

2. Find the quotient and remainder when,
   - (a) $x^3 + 4x^2 - 7x - 3$ is divided by $x - 2$
   - (b) $x^3 + 3x^2 - 4$ is divided by $x + 2$
   - (c) $3x^3 + 4x^2 - 7x + 1$ is divided by $3x - 2$

3. If $f_n(x)$ is divided by $g_m(x) \neq 0$ so that a quotient $q(x)$ and a remainder $r(x)$ are obtained, what is the degree of $q(x)$? of $r(x)$?

4. Give a linear factor of each of the polynomials.
   - (a) $r(x) = 6x^3 - 5x^2 - 17x + 6$
   - (b) $s(x) = 6x^3 - 29x^2 + 37x - 12$

5. Consider the function $f : x \rightarrow x^3 + 4x^2 + x - 6$.
   - (a) Determine $f(-3)$, $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$, and $f(3)$.
   - (b) Factor $f(x)$ over the integers.

6. If $f(x) = 2x^3 + x^2 - 5x + 2$, determine $f(x)$ at $x = -2, -1, 0, 1, 2,$ and $\frac{1}{2}$. Factor $f(x)$ over the integers.

7. If $f(x) = x^3 + 3x^2 - 12x - k$, find k so that $f(3) = 9$.

8. Find a value for k so that

$$x^3 - x^2 + kx - 12$$

is exactly divisible by $x - 3$.

9. Determine $f(1)$ if $f(-1) = 0$ and

$$f : x \rightarrow ax^5 + ax^4 + 13x^3 - 11x^2 - 10x - 2a.$$

10. (a) Divide $x^5 + x^4 - 5x^3 - x^2 + 8x - 4$ by $x - 1$.

(b) Find $g(1)$ if $g : x \to x^4 + 2x^3 - 3x^2 - 4x + 4$.

(c) Determine α, β, and γ if

$$\frac{x^5 + x^4 - 5x^3 - x^2 + 8x - 4}{(x - 1)^3} = \alpha x^2 + \beta x + \gamma.$$

(d) Determine A, B, C, D, E, F if, for all values of x,
$(x - 1)^3(x + 2)^2 = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$.

11. Consider the function $f : x \to x^3 - 3x$. We submit a table to show three successive synthetic divisions of $f(x) = x^3 - 3x$ and resulting quotients by $x - 2$.

$$\begin{array}{rrrrr}
1 & 0 & -3 & 0 & \underline{|\,2} \\
 & 2 & 4 & 2 & \\
\hline
1 & 2 & 1 & \multicolumn{1}{|r}{2} & \\
\hline
1 & 2 & 1 & & \underline{|\,2} \\
 & 2 & 8 & & \\
\hline
1 & 4 & \multicolumn{1}{|r}{9} & & \\
\hline
1 & 4 & & & \underline{|\,2} \\
 & 2 & & & \\
\hline
1 & \multicolumn{1}{|r}{6} & & &
\end{array}$$

(a) Determine $g(x)$ and $f(2)$ if

$$f(x) = (x - 2)g(x) + f(2).$$

(b) Determine $p(x)$ and $g(2)$ if

$$g(x) = (x - 2)p(x) + g(2).$$

(c) Determine $q(x)$ and $p(2)$ if

$$p(x) = (x - 2)q(x) + p(2).$$

(d) What is $q(2)$?

(e) Show that, for all x, we can write

$$f(x) = (x - 2)\{(x - 2)[(x - 2)q(2) + p(2)] + g(2)\} + f(2).$$

(f) In Chapter 2 we shall find it useful (for analysis and translation) to be able to express a given function in terms of $x - a$. We have already developed the equipment necessary to do this for a simple polynomial function such as $f : x \to x^3 - 3x$, with $a = 2$. Using the results of parts (a) through (e) of this problem determine A, B, C and D if, for all x,

$$f(x) = x^3 - 3x = A(x - 2)^3 + B(x - 2)^2 + C(x - 2) + D.$$

(g) Sketch the graph of the function

$$f : x \to x^3 - 3x.$$

(h) Sketch the graph of the function

$$F : x \to x^3 + 6x^2 + 9x + 2.$$

1-6. Locating Zeros of Polynomial Functions

From the discussion in the previous section we know that a number r is a zero of a polynomial function f if $f(r) = 0$. Furthermore we know that the zeros of first and second degree polynomial functions can be found by solving linear and quadratic equations, for which there are simple formulas.

We know how to find the zeros of polynomial functions of the first and second degree.

$$\text{If } f : x \to mx + b, \quad m \neq 0, \quad \text{then} \quad f\left(-\frac{b}{m}\right) = 0.$$

$$\text{If } f : x \to ax^2 + bx + c, \quad a \neq 0, \quad \text{then} \quad f\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right) = 0.$$

Upon examining these solutions, mathematicians noticed that the zeros are expressed in terms of the coefficients by formulas involving only the rational operations (addition, subtraction, multiplication, division) and the extraction of roots of numbers, and believed that it might be possible to express the zeros of functions of higher degree than the quadratic in the same manner. In the first half of the sixteenth century such formal expressions for the zeros of the third and fourth degree polynomial functions were obtained by Italian mathematicians. Unfortunately, these formulas are too complicated to be of practical value in mathematical analysis. Mathematicians usually find it easier even in theoretical questions to work with the polynomial rather than with any explicit expression for the zeros.

While these explorations produced some significant, if largely peripheral, results, they were later abandoned to be replaced by better procedures. Having rejected the pursuit of formulas to solve equations of higher degree, mathematicians came to believe that perhaps the most fruitful path was to guess at the solutions.

Inspection of the x-intercepts of graphs we have sketched in earlier sections enables us to approximate zeros of polynomial functions. But plotting graphs is time-consuming, and there are better methods. Inherent in the process of preparing a table for graphing, however, is information that helps us to make intelligent guesses about the zeros. This information is contained in the following theorem.

THE LOCATION THEOREM. If f is a polynomial function and if a and b are real numbers such that $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of f between a and b.

Geometrically this theorem means that the graph of f from $(a,f(a))$ to $(b,f(b))$ intersects the x-axis in at least one point.

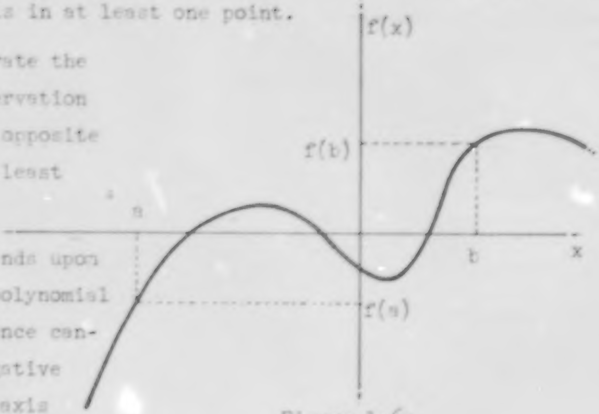


Figure 1-6a

In Figure 1-6a we illustrate the Location Theorem with the observation that $f(a)$ and $f(b)$ are of opposite sign so that f must have at least one zero between a and b.

The Location Theorem depends upon the fact that the graph of a polynomial function has no "gaps", and hence cannot have both positive and negative values without crossing the x-axis in between. A complete proof of this makes use of a suitable formulation of the fact that the real line has no "gaps" and will be discussed further in the appendices. Since the Location Theorem seems intuitively plausible we shall assume that it is true and concentrate on its consequences.

Example 1-6a. Given that the polynomial function

$$f : x \to 12x^3 - 8x^2 - 21x + 14$$

has three real zeros, locate each of them between two consecutive integers.

We use the Location Theorem to search for values of $f(x)$ that are opposite in sign. It is convenient to do this in a systematic way by synthetic substitution, setting down the work as in Table 1-6a.

Table 1-6a

Locating the Zeros of $f : x \to 12x^3 - 8x^2 - 21x + 14$

12	-8	-21	14		
12	-8	-21	14	0	
					← Location of a zero
12	4	-17	-3	1	
					←
12	16	11	36	2	
12	28	63	203	3	
12	-8	-21	14	0	
12	-20	-1	15	-1	
					←
12	-32	43	-72	-2	
			f(x)	x	

The intervals that contain the real zeros of f are indicated by the arrows at the right in the table. Thus, we see that the real zeros of f are located between 0 and 1, between 1 and 2, and between -2 and -1.

We hasten to add that it is entirely possible for f to have zeros between a and b when $f(a)$ and $f(b)$ have the same sign. We illustrate this possibility in Fig[illegible] 1-6d.

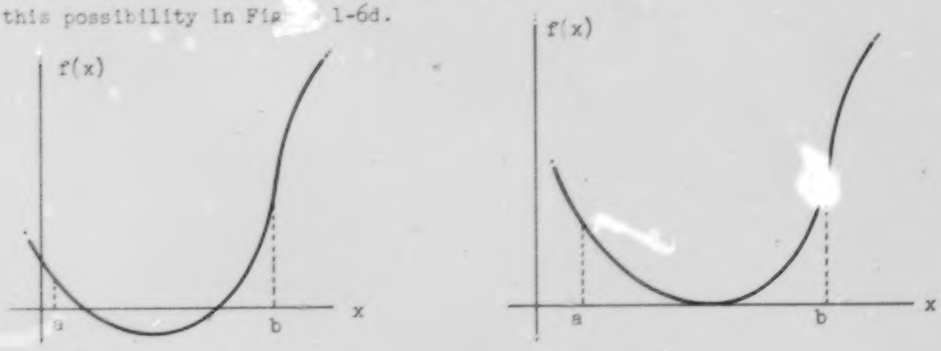


Figure 1-6d

Since the problem of locating zeros of a polynomial function is essentially a matter of trial, we should ask the very practical question: "How far should we extend the table of x and $f(x)$ when we search for the locations of the zeros of f?" In Example 1-6b this question arises.

Example 1-6b. Locate the real zeros of $f : x \rightarrow 2x^3 - x^2 - 2x + 6$.

We repeat a procedure similar to Example 1-6a and compile Table 1-6b.

Table 1-6b

Locating the Zeros of $f : x \rightarrow 2x^3 - x^2 - 2x + 6$

2	-1	-2	6	
2	-1	-2	6	0
2	1	-1	5	1
2	3	4	14	2
2	5	13	45	3
2	-1	-2	6	0
2	-3	1	5	-1
2	-5	8	-10	-2
2	-7	19	-51	-3
			$f(x)$	x

The Location Theorem tells us that there is at least one real zero r between -1 and -2. We can then write

$$f(x) = (x - r)q(x)$$

where q has degree 2, say

$$q(x) = ax^2 + bx + c.$$

Depending upon the sign of $b^2 - 4ac$, this will have two distinct real zeros, one repeated real zero or two complex zeros. Thus there are four possibilities:

(1) there may be one, two, or three real zeros, all contained in the interval between -1 and -2,

(2) two zeros may be complex, in which case there is only one real zero,

(3) one or two real zeros may be in some other interval of the table between successive integral values of x, or

(4) one or two real zeros may be in intervals outside the values of x shown in the table.

While it is difficult to rule out the possibility of complex zeros or zeros between other entries of the table we can at least show that no roots can occur outside the interval $-2 < x < 2$.

Possibility (4) appears unlikely for the simple reason that when we evaluated $f(2) = 14$, all the entries in the corresponding row of Table 1-6b were positive. They will be still greater for greater values of x; the table shows this for $x = 3$, and you can check it yourself for $x = 4$. Thus it appears that for $x > 2$, $f(x)$ must be positive, so that there cannot be a zero of f greater than 2. We shall prove this, as well as the fact that there cannot be a zero of the given polynomial less than -2, by application of the following theorem.

UPPER BOUND THEOREM FOR THE ZEROS OF A POLYNOMIAL FUNCTION. Suppose f is a polynomial function with $f(a) > 0$ for $a > 0$. If $f(x) = (x - a)q(x) + f(a)$ and if all the coefficients of $q(x)$ are positive, then all the real zeros of f are less than a. We then call a an upper bound for the zeros of f.

Proof. For $x = a$, $f(x) = f(a) > 0$. For $x > a$, by hypothesis, $x - a$, $q(x)$, and $f(a)$ are all positive. Thus, $x \geq a$ is not a zero of f, and all real zeros of f must be less than a.

Now you will see from Table 1-6b that 2 is an upper bound of the zeros of the given polynomial. We really did not need to evaluate $f(3)$.

What about a lower bound for the zeros? Since any negative root of $f(x) = 0$ is a positive root of $f(-x) = 0$, if we find an upper bound for the positive roots of $f(-x) = 0$, its negative will be a lower bound for the negative roots of $f(x) = 0$. Let us apply this test to our example.

From the given polynomial

$$f(x) = 2x^3 - x^2 - 2x + 6,$$

we find that

$$f(-x) = -2x^3 - x^2 + 2x + 6.$$

Since we are trying to find the roots of the equation, $f(-x) = 0$, it will be less confusing to multiply each member of this equation by -1 in order to have a positive coefficient for the third degree term. Thus we have

$$-f(-x) = 2x^3 + x^2 - 2x - 6.$$

Using synthetic substitution, we obtain the results shown in Table 1-6c for positive values of x.

Table 1-6c

Evaluating $-f(-x) = 2x^3 + x^2 - 2x - 6$

2	1	-2	-6	
2	1	-2	-6	0
2	3	1	-5	1
2	5	8	10	2
2	7	19	51	3
			$-f(-x)$	x

This table tells us two things. First, a positive root of $-f(-x) = 0$ occurs between 1 and 2, which means that a negative root of $f(x) = 0$ occurs between -1 and -2, as previously shown in Table 1-6b. Secondly, 2 is an upper bound for the roots of $-f(-x) = 0$, and hence, -2 is a lower bound for the roots of $f(x) = 0$. This is the conclusion for which we have been searching. In actual practice, however, it is unnecessary to evaluate $-f(-x)$ to find a lower bound for the zeros of f. Notice in Table 1-6b that the synthetic substitution for $x = -2$ gives alternating signs for the coefficients of $q(x)$ and $f(-2)$.

Suppose a negative number a is substituted (synthetically) in $f(x)$. If the coefficients of $q(x)$ concluding with the number $f(a)$ alternate in sign, then all of the real zeros of f are greater than a. We say that a is a lower bound for the zeros.

In Example 1-6b, we have found that 2 is an upper bound and -2 is a lower bound for the real zeros of the given function. Hence, all the real zeros of f are contained in the interval $-2 < x < 2$ and we have found that one zero lies between -1 and -2.

Methods for showing that, in fact, f has only one real zero (which we know must lie in the interval $-2 < x < -1$) are beyond the scope of this section.

Exercises 1-6

1. Find intervals between consecutive integers that contain the real zeros of f, given that:

   (a) $f(x) = x^3 - 3x^2 + 3$

   (b) $f(x) = 3x^3 + x^2 + x - 3$

   (c) $f(x) = 9 - x - x^2 - x^3$

   (d) $f(x) = 3x^3 - 3x + 1$ (Hint: evaluate $f(\frac{1}{2})$.)

   (e) $f(x) = 2x^3 - 5x^2 - x + 5$

   (f) $f(x) = x^3 - 3x^2 + 6x - 9$

   (g) $f(x) = x^4 - 6x^3 + x^2 + 12x - 6$

2. Determine the values of k for which $f(x) = x^3 - 2x^2 + 3x - k$ has at least one real zero between

   (a) 0 and 1

   (b) 1 and 2

3. In Example 1-6b we located at least one zero of

$$f : x \rightarrow 2x^3 - x^2 - 2x + 6$$

   between -2 and -1 (Table 1-6b). While that example served primarily as a vehicle for the development of larger considerations we afford you the satisfaction of completing it here.

   (a) Evaluate $f(-\frac{3}{2})$

   (b) Divide $2x^3 - x^2 - 2x + 6$ by $x + \frac{3}{2}$.

   (c) For what values of x does $2x^2 - 4x + 4 = 0$?

   (d) How many times does the graph of $f : x \rightarrow 2x^3 - x^2 - 2x + 6$ cross the x-axis?

   (e) How many real zeros has the function $f : x \rightarrow 2x^3 - x^2 - 2x + 6$?

   (f) What are the zeros of $f : x \rightarrow 2x^3 - x^2 - 2x + 6$?

4. (a) Locate real zeros of each of the following functions.

(i) $f : x \to x^3 - 3x$

(ii) $F : x \to x^3 + 6x^2 + 9x + 2$

(b) Factor $x^3 - 3x$ over the reals.

(c) What are the zeros of each of the following? (Consult Exercises 1-4, No. 11.)

(i) $f : x \to x^3 - 3x$

(ii) $F : x \to x^3 + 6x^2 + 9x + 2$

5. Use the Factor Theorem to find a cubic equation whose roots are -2, 1, and 3.

You are familiar with the fact that for the general quadratic equation $ax^2 + bx + c = 0$, the sum of the roots is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$. Similar relationships exist between the roots and the coefficients of polynomials of higher degree. The following problems (Nos. 6, 7, and 8) are intended to illustrate these relationships for third-degree polynomials.

6. Use the roots of the equation given in Number 5 for each of the following parts:

(a) Find the sum of the roots. Compare this result with the coefficient of x^2 obtained in Number 5.

(b) Find the sum of all possible two-factor products of the roots. That is, find $(-2)(1) + (-2)(3) + (1)(3)$. Compare this result with the coefficient of x obtained in Number 5.

(c) Find the product of the roots. Compare this result with the constant term obtained in Number 5.

7. If the roots of a 3rd-degree polynomial equation are -2, $\frac{1}{2}$, and 3, find

(a) the sum of the roots,

(b) the sum of all possible two-factor products of the roots,

(c) the product of the roots.

(d) Using the results of (a), (b), and (c), write a polynomial equation of 3rd degree having the given roots.

(e) Check your results by using the Factor Theorem to obtain the equation.

8. (a) Using the Factor Theorem, write in expanded form a 3rd-degree polynomial equation having the roots r_1, r_2, and r_3.

(b) From the result obtained in part (a), and from the fact that any polynomial of 3rd degree can be written in the form

$$a_3\left(x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3}\right),$$

find expressions for the coefficients $\frac{a_2}{a_3}$, $\frac{a_1}{a_3}$, and $\frac{a_0}{a_3}$ in terms of the roots r_1, r_2, and r_3.

9. Find the polynomial function f of degree three that has zeros -1, 1, and 4 and satisfies the condition $f(0) = 12$.

10. There is a theorem known as Descartes' Rule of Signs that states that the number of positive roots of $f(x) = 0$ cannot exceed the number of variations in sign of the coefficients of $f(x)$. A variation in sign occurs whenever the sign of a coefficient differs from the sign of the next nonzero coefficient. Thus $x^4 - x^3 + 2x + 5$ has 2 variations in sign.

Since the roots of $f(-x) = 0$ are the negatives of the roots of $f(x) = 0$, the number of negative roots of $f(x) = 0$ cannot exceed the number of variations in sign of the coefficients of $f(-x)$. Thus $f(x) = x^4 - x^3 + 2x + 5$ has at most 2 negative roots, since $f(-x) = x^4 + x^3 - 2x + 5$ has 2 variations in sign.

Find the maximum number of positive and negative roots of each of the following equations.

(a) $x^3 - x^2 - 14x + 24 = 0$

(b) $x^7 - x^4 + 3 = 0$

(c) $3x^4 + x^2 - 2x - 3 = 0$

(d) $x^5 - 1 = 0$

(e) $x^5 + 1 = 0$

(f) $x^5 = 0$

1-7. Rational Zeros

If $f(x)$ is a polynomial $a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$, all of whose coefficients $a_n, a_{n-1}, \dots, a_0$ are integers, then we may find all rational zeros of f by testing only a finite number of possibilities, as indicated by the following theorem.

> If the polynomial
>
> (1) $$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$
>
> has integer coefficients $a_n, a_{n-1}, \dots, a_0$, and if f has a rational zero $\frac{p}{q} \neq 0$, $q > 0$, expressed in lowest terms (that is, p and q are integers with no common integer divisor greater than 1), then p is a divisor of a_0 and q is a divisor of a_n.

We use the following argument to establish the theorem.

If $\frac{p}{q}$ is a zero of f, then $f(\frac{p}{q}) = 0$. By Equation (1)

$$f(\tfrac{p}{q}) = a_n(\tfrac{p}{q})^n + a_{n-1}(\tfrac{p}{q})^{n-1} + \dots + a_1(\tfrac{p}{q}) + a_0 = 0,$$

or, when cleared of fractions,

(2) $$a_n p^n + a_{n-1}p^{n-1}q + \dots + a_1 p q^{n-1} + a_0 q^n = 0.$$

Solving Equation (2) for $a_0 q^n$ we obtain

$$\begin{aligned} a_0 q^n &= -[a_n p^n + a_{n-1}p^{n-1}q + \dots + a_1 p q^{n-1}] \\ &= -p[a_n p^{n-1} + a_{n-1}p^{n-2}q + \dots + a_1 q^{n-1}] \\ &= pN, \end{aligned}$$

where $N = -[a_n p^{n-1} + a_{n-1}p^{n-2}q + \dots + a_1 q^{n-1}]$ is an integer. Hence p divides $a_0 q^n$ a whole number, N, of times. We wish to show that p divides a_0. To do this, we appeal to the Fundamental Theorem of Arithmetic, that the factorization of positive integers is unique; namely, we note that since p and q have no common integer divisor greater than 1, neither have p and

q^n. Hence, all the factors of p are factors of a_0, and p is a factor of a_0.

To prove that q divides a_n, we write Equation (2) in the form

$$(3) \qquad a_n p^n = -q[a_{n-1}p^{n-1} + \dots + a_1 p q^{n-2} + a_0 q^{n-1}].$$

Then we reason that since q divides the right-hand side of (3), it divides the number $a_n p^n$. Again, since p and q have no common divisor greater than 1, neither have q and p^n. Hence, all the factors of q are factors of a_n, and q is a factor of a_n.

The foregoing result may be easier to remember if we state it in words: If a fraction in lowest terms is a root of a polynomial equation with integer coefficients, then the numerator of the fraction must divide the constant term of the polynomial, and the denominator must divide the coefficient of the highest power of x. To keep things straight, we can always see how the theorem works for

$$mx + b = 0, \ m \neq 0.$$

The only root is $-\frac{b}{m}$; the numerator $-b$ divides b, while the denominator m divides m.

If the polynomial has fractional coefficients, the theorem can be applied after the polynomial has been multiplied by a non-zero integer to clear of fractions, because the roots of $f(x) = 0$ and the roots of $k[f(x)] = 0$ $(k \neq 0)$ are the same.

Example 1-7a. What are the rational roots of

$$3x^3 - 8x^2 + 3x + 2 = 0?$$

It is clear that 0 is not a root. If $\frac{p}{q}$ is a rational root, in lowest terms, then

$$p \text{ divides } 2, \quad q \text{ divides } 3.$$

The possibilities are

$$p = \pm 1, \pm 2, \ q = 1, 3,$$

so that

$$\frac{p}{q} = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{1}{3}, \text{ or } \pm\frac{2}{3}.$$

We test these one by one and find that the roots of the given equation are 1, 2, and $-\frac{1}{3}$.

(Note that in the statement of Theorem A2-2a, we specified $q > 0$, so the possibilities for q are all positive. There is no point in testing both $\frac{1}{3}$ and $\frac{-1}{-3}$.)

Example 1-7b. Find the rational roots of

$$3x^4 - 8x^3 + 3x^2 + 2x = 0.$$

$$\begin{aligned} f(x) &= 3x^4 - 8x^3 + 3x^2 + 2x \\ &= x(3x^3 - 8x^2 + 3x + 2). \end{aligned}$$

Now, $$f(x) = 0$$

if and only if either

$$x = 0 .$$

or

$$3x^3 - 8x^2 + 3x + 2 = 0. \tag{4}$$

By Example 1-7b, the roots of Equation (4) are 1, 2, and $-\frac{1}{3}$. Adding the root 0, we see that the roots of $f(x) = 0$ are $0, 1, 2, -\frac{1}{3}$.

We can use our Rational Zero Theorem to establish a corollary for integral zeros.

If the polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

has integer coefficients, with the constant term $a_0 \neq 0$, and with the coefficient of the highest power of x equal to 1, then the only possible rational zeros of f are integers that divide a_0.

We establish this corollary with the following short proof.

Suppose $\frac{p}{q}$ (in lowest terms), $q > 0$ is a zero of f. Since $a_0 = f(0) \neq 0$, $\frac{p}{q} \neq 0$. By the Rational Zero Theorem, p divides a_0 and q divides 1. Therefore, $q = 1$, and $\frac{p}{q} = p$ is an integer that divides a_0.

Example 1-7c. Find the rational zeros of

$$f : x \to x^3 + 2x^2 - 9x - 18.$$

By our corollary, the possible rational zeros are integers that divide -18, namely $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$. By trial, the zeros of f are -3, -2, and 3.

After we have found one zero of a polynomial function, f we can use a special device to make it easier to find further zeros. By this device, we can cut down the number of possible zeros we have to test, and sometimes we can even use it to help us find certain irrational zeros.

We know from the Factor Theorem (Section 1-5) that a is a zero of f if and only if there is a polynomial q such that

$$f(x) = (x - a)q(x). \tag{5}$$

Since the product $(x - a)q(x)$ is zero if and only if either $x - a = 0$ or $q(x) = 0$, it follows that the set of zeros of f consists of a together with the set of zeros of q:

$$\{x : f(x) = 0\} = \{x : x = a \text{ or } q(x) = 0\}. \tag{6}$$

Moreover, the degree of q is one less than the degree of f. Thus, if we can find one zero of f, Equations (5) and (6) allow us to reduce the problem of finding the zeros of f to that of finding the zeros of a polynomial q of lower degree. Naturally we may repeat the process, with q in place of f, if we are fortunate enough to find a zero of q, say b, for then we may apply the Factor Theorem to q and write

$$q(x) = (x - b)r(x),$$

and

$$\{x : q(x) = 0\} = \{x : x = b \text{ or } r(x) = 0\}.$$

If we are successful in repeating this reduction until we have a quotient which is either linear or quadratic, we can easily finish the job by solving a linear or quadratic equation.

Example 1-7d. Find all solutions of

$$2x^3 - 3x^2 - 12x + 13 = 0. \tag{7}$$

Direct calculation shows that 1 is a solution of Equation (7). Therefore, $x - 1$ is a divisor of $2x^3 - 3x^2 - 12x + 13$. Performing the division,

$$\begin{array}{rrrr l} 2 & -3 & -12 & 13 & \quad \underline{1|} \\ & 2 & -1 & -13 & \\ \hline 2 & -1 & -13 & 0 & \end{array}$$

Thus

$$2x^3 - 3x^2 - 12x + 13 = (x - 1)(2x^2 - x - 13),$$

and the solutions of Equation (7) are 1 and the solutions of

$$2x^2 - x - 13 = 0.$$

By the quadratic formula, $\frac{(1 + \sqrt{105})}{4}$ and $\frac{(1 - \sqrt{105})}{4}$ are the additional solutions of Equation (7).

Example 1-7e. Find all zeros of

$$f : x \rightarrow 12x^3 - 8x^2 - 21x + 14.$$

This is the same function that we considered earlier in Section 1-6, Example 1-6a. At that time we found that there are zeros between 0 and 1, between 1 and 2, and between -2 and -1. Thus, we know that there are three real zeros, but we do not know whether they are rational or irrational. If all three are irrational, the best we can do is to find decimal approximations. But if at least one zero is rational, then we can obtain a function of reduced degree -- in this case a quadratic -- that will enable us to find the exact values of the remaining zeros whether rational or irrational.

If the function has a rational zero, it will be of the form $\frac{p}{q}$, and by the Rational Zero Theorem of this section the possibilities for p are ± 1, ± 2, ± 7, ± 14, and for q are $1, 2, 3, 4, 6, 12$. Thus, there appear to be a good many values of $\frac{p}{q}$ to test as possible zeros of the given function. But since we already know something about the location of the zeros, we need test only those possible rational zeros $\frac{p}{q}$ between 0 and 1, between 1 and 2, and between -2 and -1, until a zero is found.

Now the possible rational zeros between 0 and 1 are

$$\frac{p}{q} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \frac{2}{3}, \frac{7}{12}.$$

By synthetic substitution, we find that $f(\frac{1}{2}) = 3$. Since $f(0) = 14$ and $f(1) = -3$, the zero lies between $\frac{1}{2}$ and 1. Hence, we need not test the values $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}$, and $\frac{1}{12}$. This is a good example of how the Location Theorem (Section 1-6) may save us unnecessary work.

Continuing, we know that the only possible rational zero between $\frac{1}{2}$ and 1 is $\frac{2}{3}$ or $\frac{7}{12}$. Testing these, we find that $f(\frac{2}{3}) = 0$, and we have found the rational zero $\frac{2}{3}$. By the Factor Theorem, $x - \frac{2}{3}$ is a divisor of $f(x)$, and the quotient, obtained from the synthetic substitution of $\frac{2}{3}$, is

$$q(x) = 12x^2 - 21.$$

The zeros of q are the roots of

$$12x^2 - 21 = 0,$$

which are $\frac{\sqrt{7}}{4}$ and $-\frac{\sqrt{7}}{4}$. Thus, the zeros of the given polynomial are $\frac{2}{3}$, $\frac{\sqrt{7}}{4}, -\frac{\sqrt{7}}{4}$.

Exercises 1-7

Find all rational zeros of the polynomial functions in Exercises 1 - 12, and find as many irrational zeros as you can.

1. (a) $x \to 2x^2 - 3x - 2$

   (b) $x \to 2x^3 - 3x^2 - 2x$

2. (a) $x \to x^3 - 6x^2 + 11x - 6$

   (b) $x \to x^4 - 6x^3 + 11x^2 - 6x$

3. (a) $x \to x^3 - 2x^2 + 3x - 4$

   (b) $x \to x^4 - 2x^3 + 3x^2 - 4x$

4. (a) $x \to 2x^3 - x^2 - 2x + 1$

   (b) $x \to 2x^4 - x^3 - 2x^2 + x$

5. $x \to 12x^3 - 40x^2 + 19x + 21$

6. $x \to 3x^3 - 10x^2 + 5x + 4$

7. $x \to 4x^3 - 10x^2 + 5x + 6$

8. $x \to x^4 - 2x^3 - 7x^2 + 8x + 12$

9. $x \to x^4 - 8x^2 + 16$

10. $x \to x^4 - 5x^3 + 5x^2 + 5x - 6$

11. $x \to x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$

12. $x \to 3x^4 - 8x^3 - 28x^2 + 64x - 15$

13. Show algebraically that the equation $x + \frac{1}{x} = n$ has no real solution if n is a real number such that $|n| < 2$.

You are familiar with the fact that for the general quadratic equation, $ax^2 + bx + c = 0$, the sum of the roots is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$. Similar relationships exist between the roots and the coefficients of polynomials of higher degree. The following problems are intended to illustrate these relationships for third-degree polynomials.

1-8. Approximating Zeros

Methods for finding rational zeros of polynomial functions are discussed in Section 1-7. A simple method, known as the method of repeated bisection can be used, along with the Location Theorem, to approximate roots (either rational or irrational). This method is easy to describe and is simple to program on a computer. The arithmetic can become very complicated, however, and the method is fairly slow. (Another, more powerful, method is described in Section 2-10.)

Consider the polynomial function

$$f : x \to x^3 + 3x - 1.$$

Since $f(0) = -1 < 0$ and that $f(1) = 3 > 0$, we know (by the Location Theorem) that there is at least one zero between 0 and 1. We take the average of 0 and 1, namely $\frac{1}{2}$, and find

$$f(\tfrac{1}{2}) = \frac{1}{8} + \frac{3}{2} - 1 > 0.$$

Thus, there is a zero between 0 and $\frac{1}{2}$. We average again to obtain $\frac{1}{4}$. Since

$$f(\tfrac{1}{4}) = \frac{1}{64} + \frac{3}{4} - 1 < 0,$$

we know that there is a zero between $\frac{1}{4}$ and $\frac{1}{2}$. Averaging these we get

$$\frac{\frac{1}{4} + \frac{1}{2}}{2} = \frac{3}{8}$$

and since

$$f(\tfrac{3}{8}) = \frac{27}{512} + \frac{9}{8} - 1 > 0,$$

we have located a zero between $\frac{1}{4}$ and $\frac{3}{8}$. For convenience we now use decimal notation and average again to obtain

$$\frac{0.25 + 0.375}{2} = 0.3125.$$

Since $f(0.3125) < 0$ we know that there is a zero between 0.3125 and 0.375.

We could continue the process to show (Exercises 1-8, No. 1) that the zero is between 0.31 and 0.35. Having done this we could be certain that the zero is 0.3, correct to one decimal place.

For obtaining zeros of functions to some prescribed degree of accuracy the method of repeated bisection is often used on high speed computers since the process is easy to program.

Without a computer we try to speed the process by shrewd guessing. For example, we might observe that

$$x^3 + 3x - 1$$

is positive for $x > \frac{1}{3}$. For $x > \frac{1}{3}$, we have

$$x^3 + 3x - 1 > (\frac{1}{3})^3 + 3(\frac{1}{3}) - 1 > 0.$$

We could then test 0.333, 0.332, 0.331, 0.330, 0.329, etc. until we obtain a negative value; and then average to obtain further accuracy. Does this really speed the process?

Exercises 1-8

1. Show that a zero of $f : x \to x^3 + 3x - 1$ lies between 0.3 and 0.4.

2. Extend the method of repeated bisection started in this section for the function

$$f : x \to x^3 + 3x - 1$$

(a) to locate a zero of f between 0.31 and 0.35;

(b) to show that a zero of f, correct to two decimal places, is 0.32.

3. Find correct to the nearest 0.5, the real zero of $f : x \to x^3 - 3x^2 - 2x + 5$ that lies between 3 and 4.

4. (a) Find, correct to the nearest 0.5, the real zeros of $f : x \to x^3 - 2x^2 + x - 3$.

(b) Find the zeros correct to the nearest 0.1.

5. (a) Find a solution of $x^3 + x = 3$ correct to one decimal place.

(b) Find this solution correct to two decimal places.

6. Find the real cube root of 20 correct to two decimal places by solving the equation $x^3 = 20$.

1-9. Degree of Polynomial and Behavior of Graph

Suppose f is the polynomial function

$$f : x \rightarrow a_0 + a_1x + a_2x^2 + \ldots + a_nx^n, \quad a_n \neq 0.$$

What kinds of information about the graph of f can we get easily from this expression? For example, note that

$$f(0) = a_0$$

so that the constant term a_0 is the y-intercept, that is the graph crosses the y-axis at the point $(0,a_0)$.

This observation is, of course, quite simple. In the next chapter we shall show that the coefficient a_1 is the slope of the tangent line to the graph of f at $(0,a_0)$. The other coefficients of f will also be of significance as we try to determine the behavior of f near the point $(0,a_0)$.

The degree of f can also give us useful information.

Suppose we wish to know how many times a line given by $g(x) = mx + b$ can intersect the graph of a polynomial function

$$f : x \rightarrow a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0.$$

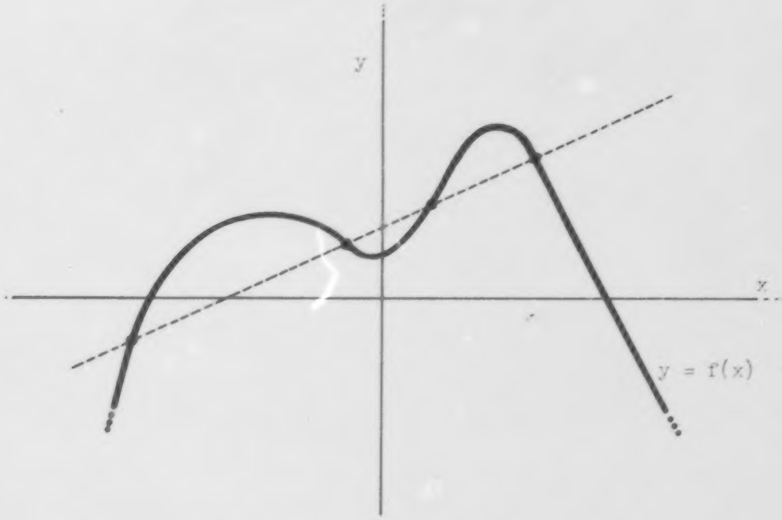


Figure 1-9a

That is, for how many values of x is it possible that $f(x) = g(x)$? In other words, we are asking how many roots the equation $f(x) - g(x) = 0$ can have, or the maximum number of zeros of the function

$$F : x \to a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + (a_1 - m)x + (a_0 - b).$$

We assume (without proof here):

(1) if f is a polynomial function of degree $n > 0$, then f has at most n real zeros.

Since F is of the same degree as f we know that F has at most n real zeros. This means that a line can intersect a polynomial curve no more times than the degree of the polynomial.

The x-axis is a very special case of a line given by $y = mx + b$, where m and b are both zero. Therefore, as a particular consequence of (1) we have:

(2) if f has degree $n > 0$, then the graph of f can cross the x-axis no more than n times.

The expression for $f(x)$ also determines the behavior of f for values of x far from the origin. For example, consider the function

$$f : x \to 1 - 3x + 2x^2 + x^3.$$

If x is far from zero (that is, $|x|$ is large) then the cubic term x^3 dominates the remaining terms. To show this we can rewrite the expression for $f(x)$ as

$$1 - 3x + 2x^2 + x^3 = x^3\left(\frac{1}{x^3} - \frac{3}{x^2} + \frac{2}{x} + 1\right)$$

as $|x|$ increases, the absolute value of each of the terms

$$\frac{1}{x^3}, \quad \frac{-3}{x^2}, \quad \text{and} \quad \frac{2}{x}$$

decreases so that

$$\frac{1}{x^3} - \frac{3}{x^2} + \frac{2}{x} + 1 \text{ is close to } 1$$

when $|x|$ is very large.

By this kind of reasoning, we could show that for any polynomial function f, the term of highest degree will dominate all other terms when x is far from zero. This means that the sign of f(x) will agree with the sign of the term of highest degree for |x| large, and hence the graph of f will lie above or below the x-axis according as the value of this term is positive or negative.

We combine this information with that previously garnered to sketch the possible graph of

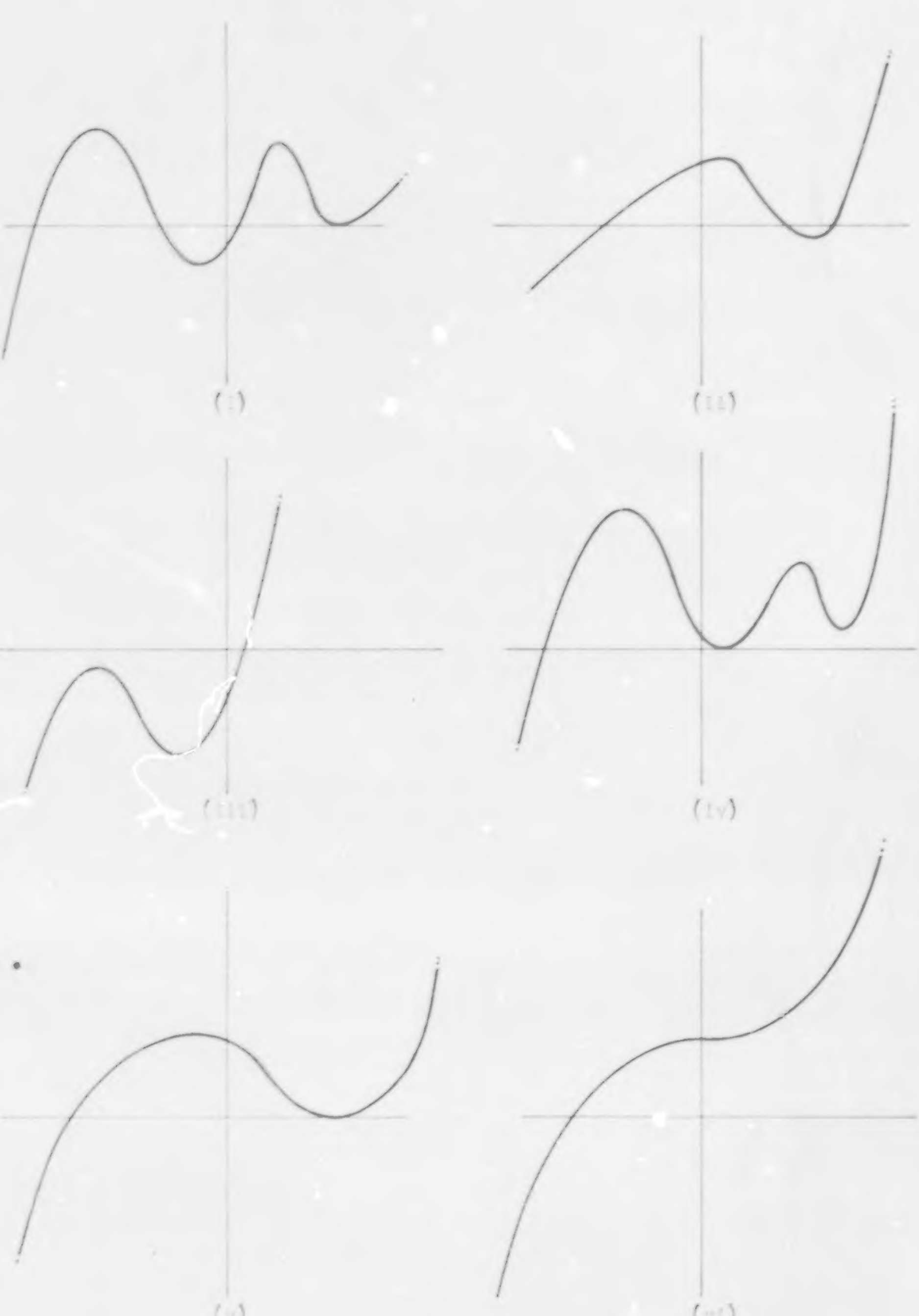
$$f : x \to 1 - 3x + 2x^2 + x^3.$$

We know that f(0) = 1, and that the graph of f can cross any line, and in particular the x-axis, at most three times. Furthermore, the term x^3 dominates when |x| is large, so that for x far to the right the graph of f must be far above the x-axis and for x far to the left, the graph of f lies far below the x-axis. In particular, the graph of f must cross the x-axis to the left of the origin (since f(x) < 0 for x far left and f(0) > 0). Some candidates for the graph of f are sketched in Figure 1-9b. Further information is needed to show which graph might be an accurate picture of f. In the next chapter we shall develop methods for determining the behavior of graphs of polynomial functions (e.g., locating maximum and minimum points). For now we can eliminate five of the six possibilities pictured. We eliminate:

(i) and (iv) because no line should be able to cross the graph more than three times;

(iii) because f(0) must be positive;

(ii) and (vi) because f(-1) must be greater than f(0).



(i) (ii) (iii) (iv) (v) (vi)

Figure 1-9b

Candidates for the graph of $x \to 1 - 3x + 2x^2 + x^3$

Exercises 1-9

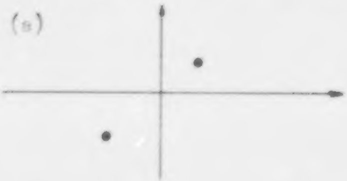
1. Plotting as few points as possible try to sketch the graph of $y = 3x^4 + 4x^3 - 12x^2 + 5$.

2. Plotting as few points as possible try to sketch the graph of $f : x \to x^5 + x^4 - 5x^3 - x^2 + 8x - 4$.

3. "If f and g are polynomial functions and $f(x) = 0$ if and only if $g(x) = 0$, then f and g are identical polynomial functions." Refute or defend this statement.

4. Suppose that there are only a finite number of selected points shown for a number of polynomial functions. One could only guess at the complete graph. In each case indicate the minimum degree that a polynomial function might have and still be satisfied by these points.

(a)



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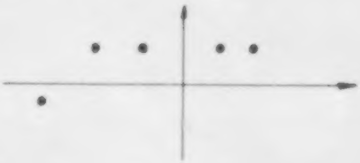

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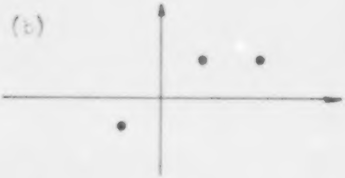

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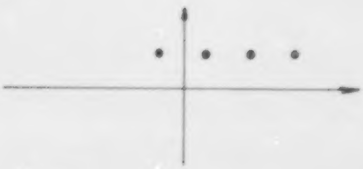

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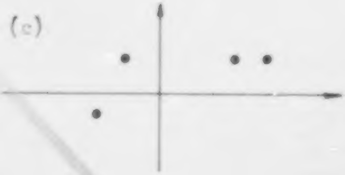
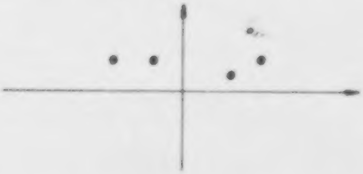
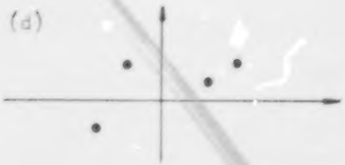


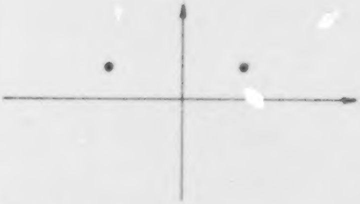
(h)



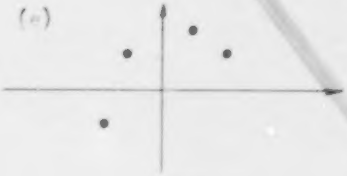
(d)



(i)



(e)



5. Suppose that f is a polynomial function of degree n and $g : x \to f(ax + b)$, where a and b are constants, $a \neq 0$.

   (a) Is g a polynomial function? If so what is its degree? If not why not?

   (b) If $a = 1$ how is the graph of g related to the graph of f?

   (c) If $b = 0$ how are the graphs related?

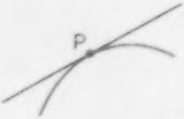
   (d) Use parts (b) and (c) to indicate the relationship between the graphs of f and g for general a and b. Consider $g(x) = f(a(x + \frac{b}{a}))$ and $g(x) = f(ax + b)$.

Chapter 2

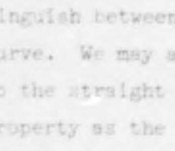
THE DERIVATIVE OF A POLYNOMIAL FUNCTION

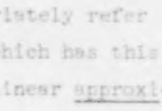
Having discussed polynomial functions in Chapter 1 we now turn to one aspect of the calculus of polynomial functions. The two basic ideas of the elementary calculus are derivative and integral. We can appreciate these ideas intuitively and understand their usefulness before we formulate them precisely. We begin with the idea of derivative.

If we select any point P on the graph of a polynomial function and draw a line through P with a ruler, it will be possible to choose the direction of the ruler so that very close to P the line seems to lie along the graph. When this is done, if we stay close enough to P, it will be impossible to distinguish between the line and the curve. We may appropriately refer to the straight line which has this property as the best linear approximation of the graph at P. The straight line is also said to touch or be tangent to the graph at P. In this chapter, we shall be concerned with the precise determination of the direction of the tangent line at any point of a polynomial graph.



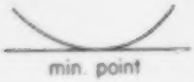
Note that our use of the word "tangent" here is consistent with its meaning in the elementary geometry of circles, but it is also more inclusive. Graphs of polynomial functions may lie entirely on one side or the other of one of their tangents, as circles do, but they may also cross their tangents:

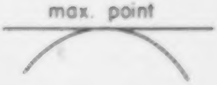




The derivative will help us to determine the direction of such tangents, and also the shape of the curve.

Once we know how to determine the tangent and the shape of the graph we shall be in a position to find any points on a polynomial graph at which the tangent line is horizontal and the graph nearby is entirely above or entirely below the tangent. Such points are called minimum and maximum points, respectively.





The problem of finding the tangent to a polynomial graph at a point P and the shape of the graph nearby is particularly simple if the point is on the y-axis. As we shall see, in this case the result can be written by inspection. At first we shall, therefore, confine ourselves to this easy special case, and later turn to the case in which the point is not on the y-axis.

From these considerations we shall obtain a general formula for the slope of the tangent to the graph of a polynomial function f at any point $(x,f(x))$. The general result will be expressed as a new function, derived from f, sometimes thought of as the slope function of f. It is this slope function which we call the derivative of the function f. In the final sections of this chapter we shall apply these ideas as we examine the behavior of polynomial functions and in later chapters we shall see that the same basic concepts can be used to discuss functions other than polynomial functions.

2-1. The Tangent at the y-Intercept of a Graph

In this section we shall illustrate the method of obtaining an equation of the tangent to a polynomial graph at its point of intersection with the y-axis. As indicated in the introduction, the tangent we are seeking is defined here to be the straight line most closely approximating the curve at a given point.

For a polynomial the method consists merely of omitting every term whose degree is higher than one.

Example 2-1a. The graph G of $f : x \to 1 + x - 4x^2$ intersects the y-axis at P(0,1). The tangent T to G at P has the equation

$$y = 1 + x$$

obtained by omitting the second degree term $-4x^2$. It is easy to draw T from its equation.

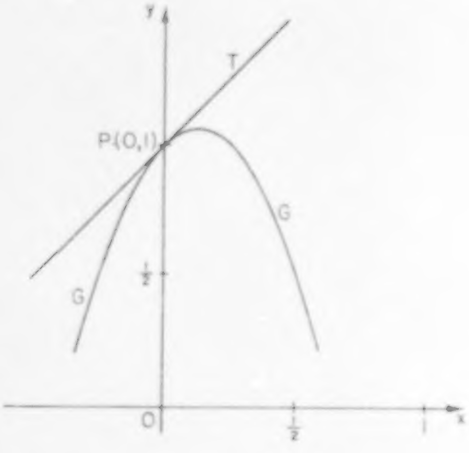


Figure 2-1a

G is the graph of $f : x \to 1 + x - 4x^2$

T is the graph of $y = 1 + x$

Moreover, since the omitted term $-4x^2$ is negative for all values of x except 0, G lies below T except at P. (See Figure 2-1a.)

Example 2-1b. The graph G of $f : x \to 2 + x^2$ intersects the y-axis at P(0,2). If we omit the x^2 term and write $y = 2$ we obtain the equation of the tangent T through P. In this case the tangent is parallel to the x-axis. Since x^2 is positive for all x except zero, all points of G except P lie above the tangent line T.

Because P is the lowest point on G, it is called the minimum point of the graph. (See Figure 2-1b.)

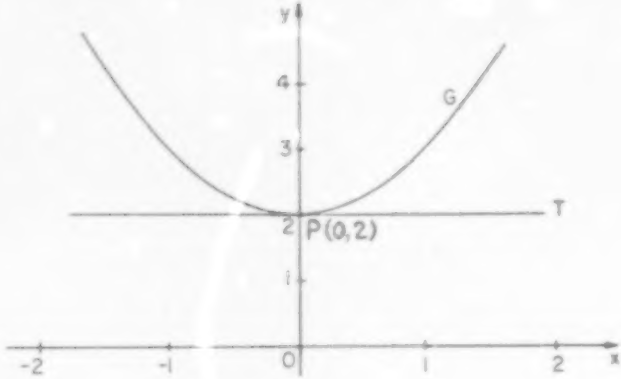


Figure 2-1b

G is the graph of $f : x \to 2 + x^2$

T is the graph of $y = 2$

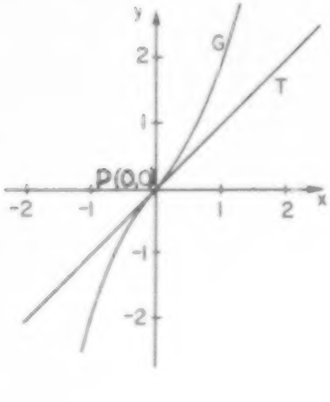


Figure 2-1c

G is the graph of $f : x \to x + x^3$

T is the graph of $y = x$

Example 2-1c. The graph of $f : x \to x + x^3$ intersects the y-axis at $P(0,0)$. The equation

$$y = x$$

of the tangent at P is obtained by omitting the x^3 term. Since x^3 is positive for positive x and negative for negative x, G is above T if $x > 0$ and below T if $x < 0$. (See Figure 2-1c.) The graph G therefore crosses from one side of the tangent to the other. P is called a point of inflection of the graph G.

The pictures for Examples 2-1a, b, and c seem to indicate that the procedure of omitting every term whose degree is higher than one does indeed produce the equation of the tangent to a polynomial graph at its y-intercept. To explain why, we return to Example 2-1a. We obtained the equation $y = 1 + x$ of the tangent to the graph of

(1) $$f : x \to 1 + x - 4x^2$$

at $(0,1)$ by omitting the term $-4x^2$. We wish to justify this procedure by showing that the line obtained does represent the best linear approximation to the graph at the point P. This will entitle us to call $y = 1 + x$ the equation of the tangent to the graph at P.

From (1) we have

$$f(x) = 1 + x - 4x^2,$$

which may be written as

(2) $$f(x) = 1 + (1 - 4x)x.$$

If x is numerically small, the expression $1 - 4x$ in parentheses is close to 1. In fact, we can make $1 - 4x$ lie as close to 1 as we please by making x numerically small.

Specifically, if we wish $1 - 4x$ to be within $.01$ of 1 and hence to lie between $.99$ and 1.01, it will be sufficient to make $4x$ lie between $-.01$ and $.01$, and therefore to make x lie between $-.0025$ and $.0025$.

This result has a simple geometrical interpretation (see Figure 2-1d). Let us consider three lines L, L_1, and L_2 through $P(0,1)$ with slopes 1, $1 + .01$ and $1 - .01$. These lines have the equations

$$L : y = 1 + x$$
$$L_1 : y = 1 + 1.01x$$
$$L_2 : y = 1 + .99x$$

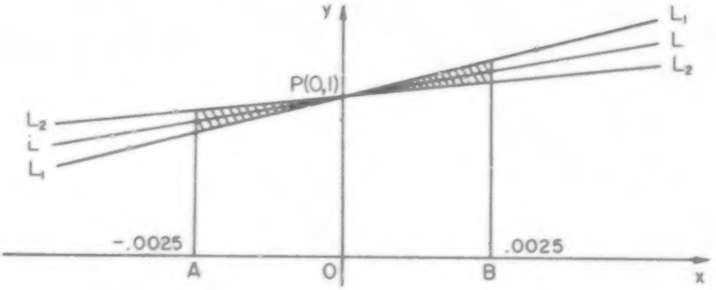


Figure 2-1d

Their slopes are so nearly equal that the differences can be shown on Figure 2-1d only by distorting the scale. Let AB be the interval $-.0025 < x < .0025$. On this interval AB, the graph of $f : x \to 1 + (1 - 4x)x$ lies between L_1 and L_2 and, hence, in the hatched region.

The numbers chosen were merely illustrative. They were designed to give a certain concreteness to the picture. We can make $1 - 4x$ lie between $1 + \epsilon$ and [illegible] for an arbitrarily small value of ϵ, merely by choosing x between [illegible] and $\frac{\epsilon}{4}$. We did not need to choose $\epsilon = .01$.

Geometrically this means that if we keep values for x close enough to zero the graph of $f : x \to 1 + (1 - 4x)x$ lies between two lines.

$$L_1 : y = 1 + (1 + \epsilon)x$$

$$L_2 : y = 1 + (1 - \epsilon)x$$

which differ in direction as little as we please. The only straight line which is always included between such lines L_1 and L_2 is

$$L : y = 1 + x.$$

Hence, we see that L can indeed be regarded as the best linear approximation to $f : x \to 1 + x - 4x^2$ at $x = 0$.

We can confine the graph G of $f : x \to 1 + x - 4x^2$ to a smaller part of the hatched region in Figure 2-1d by noting that G lies below L except at the point P. Hence, on the interval AB, G lies between L and L_2 to the right of P and between L and L_1 to the left of P. (See Figure 2-1e.)

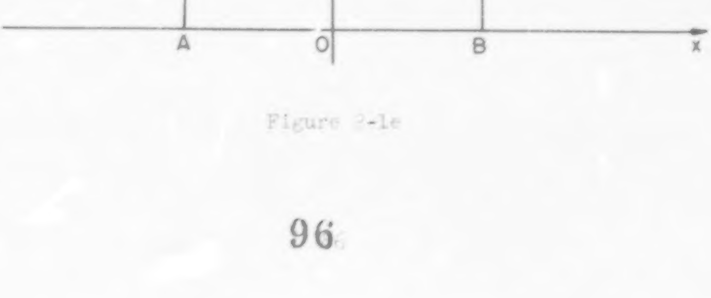


Figure 2-1e

Exercises 2-1

1. For each of the following

   (i) write the equation of the tangent to the graph of the function at the point of intersection of the graph of the function with the y-axis;

   (ii) draw the tangent line and sketch the shape of the graph near its y-intercept.

   (a) $x \to 1 - x + x^2$

   (b) $x \to 4 - x^2$

   (c) $x \to 2 + 3x - 2x^2$

   (d) $x \to 3 + 2x + x^4$

   (e) $x \to 1 + x + x^3$

   (f) $x \to 1 - x + x^3$

   (g) $x \to 2 - x^3$

   (h) $x \to 1 + 2x + x^5$

   (i) $x \to x + x^5$

   (j) $x \to x^4$

2. (a) For $f : x \to 1 + x + x^2$ show that if $-.01 < x < .01$, then $1 + .99x < f(x) < 1 + 1.01x$.

   (b) Strengthen the result of part (a) by showing that

   (i) $1 + x < f(x) < 1 + 1.01x$, for $x > 0$

   (ii) $1 + x < f(x) < 1 + .99x$, for $x < 0$.

   Show the improved results on a diagram.

   (c) Show that the results of part (a) can be obtained more simply by noticing that except at the y-intercept, the graph of $f : x \to 1 + x + x^2$ must lie above the graph of $y = 1 + x$.

3. In Example 2-1c we could write $f : x \to x + x^3$ as $f : x \to (1 + x^2)x$.

   (a) Show that

   (i) $x < f(x) < 1.01x$, for $0 < x < .1$

   (ii) $1.01x < f(x) < x$, for $0 > x > -.1$

   (b) Draw a figure to illustrate the geometrical meaning of the results in (a) and (b).

4. Consider the function $f : x \to 2 + 3x - x^2$.

   (a) At what point does the graph of the function cross the $f(x)$ axis?

   (b) Show that if $|x| < .01$,

   $$3.01 > 3 - x > 2.99$$

   and that $f(x)$ lies between

   $$2 + 3.01x \quad \text{and} \quad 2 + 2.99x.$$

   (c) Draw a figure to illustrate the geometrical meaning.

5. Strengthen the result of Number 4 by noticing that the graph of the function lies below the graph of the straight line

   $$y = 2 + 3x.$$

   What additional refinement can be made in the figure associated with Number 4?

6. Consider the function

   $$f : x \to x^2 - 2x - 1 = -1 + (-2 + x)x.$$

   (a) Show that if $0 < x < .01$, then the graph of f lies between the lines whose equations are

   $$y = -1 - 2.01x$$

   and

   $$y = -1 - 1.99x.$$

   (b) Draw a figure to show the geometrical interpretation of this result.

7. Consider the function $f : x \to 3 - 5x - 4x^2$.

   (a) For $-.02 < x < .02$ determine the slopes of the lines between which the graph of f lies near the point $(0,3)$.

   (b) If it is desired that near $(0,3)$ the graph of f lies between the straight lines $y = 3 - 4.998x$ and $y = 3 - 5.002x$, what values may x assume?

8. If f is a polynomial function of degree higher than 2, the best quadratic (or parabolic) approximation to f near its y-intercept is found by omitting every term whose degree is higher than two. Thus, the best quadratic approximation to

$$f : x \to 1 - x + x^2 - 2x^3$$

is

$$g : x \to 1 - x + x^2.$$

   (a) On the same set of axes draw the graphs of f and its best linear and best quadratic approximations near the y-intercept of f.

   (b) If $x = 0.1$, compute $f(x) - g(x)$.

   (c) If $x = 0.01$, compute $f(x) - g(x)$.

   (d) As x approaches zero, what value, if any, does $\frac{f(x) - g(x)}{x^2}$ approach?

9. The best cubic approximation to a polynomial function at its y-intercept is found by omitting every term whose degree is higher than three.

   (a) Determine the best linear, quadratic, and cubic approximations to $f : x \to 2 - x^3 + 2x^4 - x^5$ near its y-intercept $P(0,2)$.

   (b) Graph f near $P(0,2)$, making use of the information you can glean from its best linear, quadratic, and cubic approximations near there.

   (c) Let $g(x)$ be the value of the best cubic approximation to the graph of f near its y-intercept. As x approaches zero, what value, if any, does $\frac{f(x) - g(x)}{x^3}$ approach?

2-2. The Behavior of the Graph Near an Arbitrary Point

In Section 2-1 we restricted our attention to the behavior of the graph of a polynomial function near its y-intercept. Now we shall generalize our discussion to include the behavior of the graph of such a function near any point. In Section 2-1 the behavior near the point for which $x = 0$ was determined from the expression for $f(x)$ in ascending powers of x. The behavior near the point for which $x = a$, say, can be determined if we have an expression for $f(x)$ in ascending powers of $x - a$.

To begin we consider a specific function, $f : x \rightarrow 9 - 10x + 4x^2$ at a particular point, where $a = 1$.

Example 2-2a. Determine the behavior of the graph of the function $f : x \rightarrow 9 - 10x + 4x^2$ near the point $P(1,3)$.

Writing $f(x)$ in powers of $(x - 1)$ we find

(1) $$f(x) = 3 - 2(x - 1) + 4(x - 1)^2.$$

(Soon we shall see how to derive such an expansion for $f(x)$, but for the moment merely check that

$$3 - 2(x - 1) + 4(x - 1)^2 = 3 - 2x + 2 + 4x^2 - 8x + 4 = 9 - 10x + 4x^2 = f(x)$$

as desired.)

In this form (1) the graph of f may be interpreted as the result of translating the graph of the function $g : x \rightarrow 3 - 2x + 4x^2$ one unit to the right. (See Section 1-3.) Hence, the behavior of the graph of f near $x = 1$ is identically the same as the behavior of the graph of g near $x = 0$.

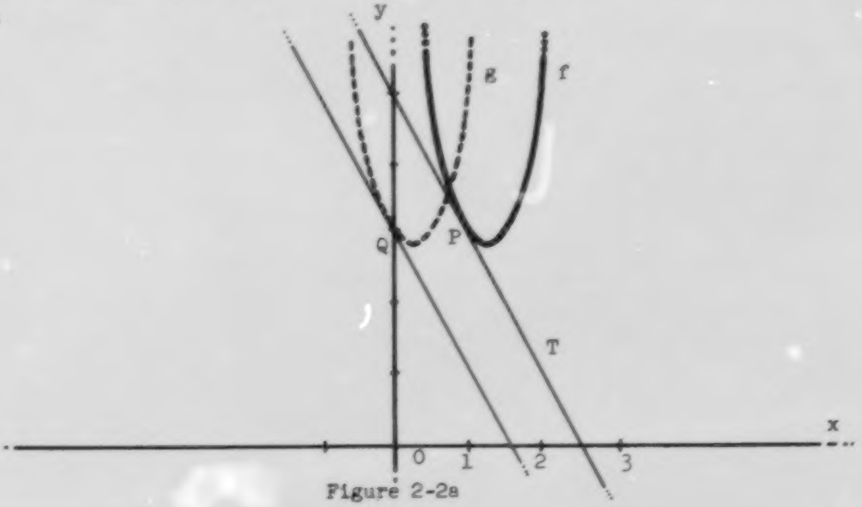


Figure 2-2a

Since the tangent to g at $x = 0$ is given by $y = 3 - 2x$, the tangent to f at $x = 1$ is given by $y = 3 - 2(x - 1)$.

Since $x^2 > 0$ for all $x \neq 0$, the graph of $g : x \rightarrow 3 - 2x + 4x^2$ lies above its tangent line $y = 3 - 2x$, except at the point of tangency $Q(0,3)$. In the same manner, since $(x - 1)^2 > 0$ for all $x \neq 1$, the graph of $f : x \rightarrow 3 - 2(x - 1) + 4(x - 1)^2$ lies above its tangent $y = 3 - 2(x - 1)$ except at the point of tangency $P(1,3)$.

The foregoing discussion assumes that since $y = 3 - 2x$ is the equation of the tangent to g at $x = 0$, then the equation of the translated line, $y = 3 - 2(x - 1)$, will represent the tangent to f at $x = 1$. Without translating the graph of g we can verify that the line given by $y = 3 + 2(x - 1)$ is the tangent to the graph of f at $x = 1$, in much the same way that we carried out the argument in Section 2-1 for tangents at the y-intercept.

Writing (1) in a factored form

$$f(x) = 3 + [-2 + 4(x - 1)](x - 1),$$

we note that if x is near enough to 1, that is if $|x - 1|$ is sufficiently small, the expression $[-2 + 4(x - 1)]$ is arbitrarily close to -2. In other words, for any ϵ, however small, $f(x)$ lies between

$$3 + (-2 + \epsilon)(x - 1)$$

and

$$3 + (-2 - \epsilon)(x - 1)$$

provided that $|4(x - 1)| < \epsilon$, that is, that $|x - 1| < \frac{\epsilon}{4}$. Hence $3 - 2(x - 1)$ is the best linear approximation to $f(x)$ near $x = 1$ and T is the tangent to the graph G at the point $P(1,3)$. It should be noted that we have followed the same procedures as before with $x - 1$ in place of x.

Thus, to describe the behavior of the graph of a function $f : x \rightarrow b_0 + b_1x + b_2x^2 + \dots + b_nx^n$ near the point where $x = a$, we need only express $f(x)$ in the form

$$c_0 + c_1(x - a) + c_2(x - a)^2 + \dots + c_n(x - a)^n.$$

The best linear approximation to f at $x = a$ is then $y = c_0 + c_1(x - a)$, the equation of the tangent to f at $x = c$. The best quadratic approximation (See Exercises 8 and 9, Section 2-1) is given by $y = c_0 + c_1(x - a) + c_2(x - a)^2$, and so on.

Now consider the problem of expanding a given function in powers of $(x - a)$ for some a. We want $f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots + c_n(x - a)^n$. Note that upon dividing $f(x)$ by $(x - a)$ the quotient is $[c_1 + c_2(x - a) + \dots + c_n(x - a)^{n-1}]$ and the remainder is c_0. Hence, to find the first coefficient c_0 of the desired expansion we divide $f(x)$ by $(x - a)$ and record the remainder. Similarly, to find c_1 we divide the quotient

$$[c_1 + c_2(x - a) + c_3(x - a)^2 + \dots + c_n(x - a)^{n-1}]$$

by $(x - a)$ again. The remainder will be c_1 and the (second) quotient will be

$$[c_2 + c_3(x - a) + \dots + c_n(x - a)^{n-2}].$$

Continuing in this manner, we can find the coefficient c_i for each power of $(x - a)$ in the expansion of $f(x)$.

For example, to expand $f(x) = 4 - 3x + 2x^2$ in powers of $(x - 1)$ we divide $f(x)$ by $(x - 1)$. By synthetic division we have

$$\begin{array}{rr|r|l} 2 & -3 & 4 & \underline{1} \\ & 2 & -1 & \\ \hline 2 & -1 & 3 & \end{array}$$

indicating that

$$f(x) = 3 + [2x - 1](x - 1). \tag{2}$$

Now we divide the quotient $(2x - 1)$ by $(x - 1)$ again:

$$\begin{array}{r|r|l} 2 & -1 & \underline{1} \\ & 2 & \\ \hline 2 & 1 & \end{array}$$

which tells us that $(2x - 1) = 1 + 2(x - 1)$.

Substituting in (2) we have

$$f(x) = 3 + [1 + 2(x - 1)](x - 1)$$

or

$$f(x) = 3 + 1(x - 1) + 2(x - 1)^2.$$

Note that the coefficients 3, 1, and 2 are precisely the remainders under repeated division by $(x - 1)$.

If $f(x)$ were an expression of higher degree, the process would be continued. We simply divide each successive quotient by $(x - 1)$ and record the remainder, until $f(x)$ is completely expressed in powers of $(x - 1)$.

Example 2-2b. Determine the behavior of the graph of $f : x \rightarrow 2 + 3x + x^2 - x^3$ near the point at which $x = 2$.

We need to expand $f(x)$ in powers of $x - 2$; that is, to find the coefficients in

$$f(x) = c_0 + c_1(x - 2) + c_2(x - 2)^2 + c_3(x - 2)^3.$$

As before, if $f(x)$ is divided by $(x - 2)$ the remainder is c_0 and the quotient is $c_1 + c_2(x - 2) + c_3(x - 2)^2$. If this quotient is divided by $x - 2$, the remainder is c_1 and the new quotient is $c_2 + c_3(x - 2)$. A further division of $c_2 + c_3(x - 2)$ by $x - 2$ gives the remainder c_2 and the final quotient c_3. We proceed to carry out these divisions synthetically.

Dividing by $x - 2$

$$\begin{array}{rrr|r l} -1 & +1 & +3 & +2 & \underline{\lfloor 2} \\ & -2 & -2 & +2 & \\ \hline -1 & -1 & 1 & 4 & \end{array}$$

We obtain the first remainder $c_0 = 4$ and the quotient

$$-x^2 - x + 1.$$

Dividing this quotient by $x - 2$

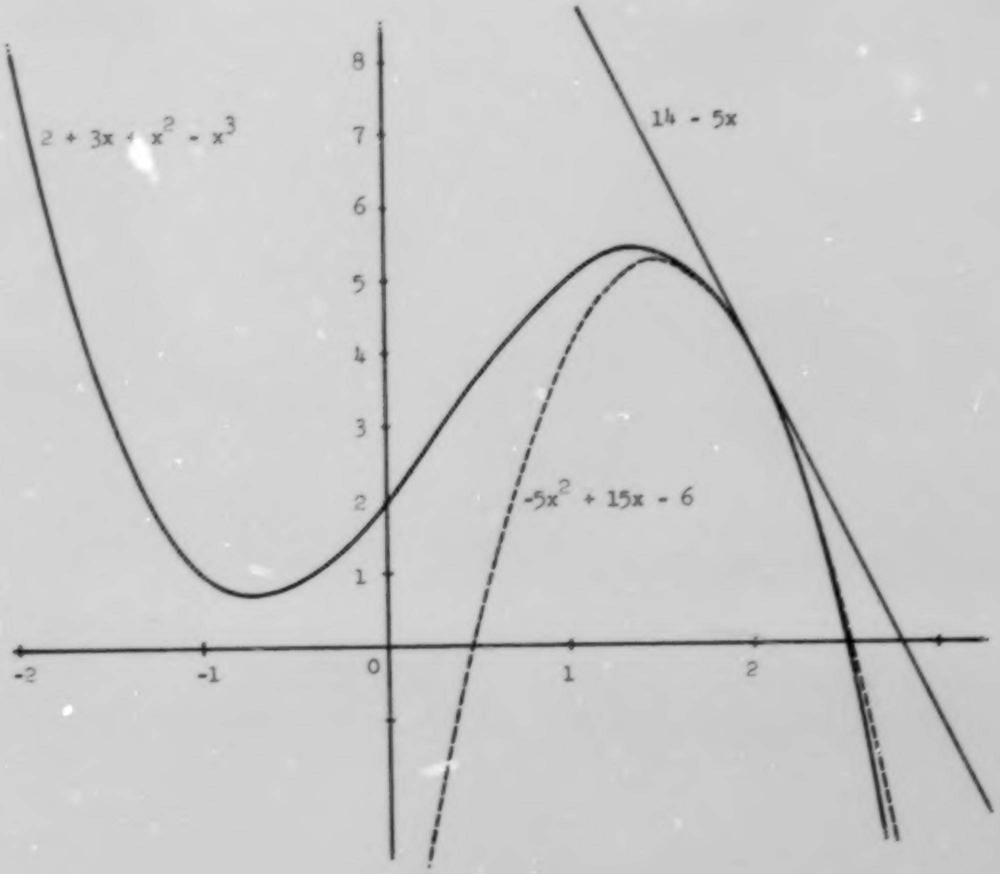
$$\begin{array}{rr|r l} -1 & -1 & +1 & \underline{\lfloor 2} \\ & -2 & -6 & \\ \hline -1 & -3 & -5 & \end{array}$$

gives the remainder $c_1 = -5$ and the new quotient $-x - 3$. Finally, dividing this quotient by $x - 2$, we have

$$\begin{array}{r|r l} -1 & -3 & \underline{\lfloor 2} \\ & -2 & \\ \hline -1 & -5 & \end{array}$$

the remainder $c_2 = -5$ and the quotient -1. The successive coefficients in the expansion of $f(x)$ in powers of $x - 2$ are the successive remainders obtained: $c_0 = 4$, $c_1 = -5$, $c_2 = -5$; the final quotient $c_3 = -1$. Thus we can write

$$f(x) = 4 - 5(x - 2) - 5(x - 2)^2 - 1(x - 2)^3.$$



Near the point where $x = 2$, we conclude that:

(1) the value of the function is 4;

(2) the equation of the best linear approximation to the graph of f is $y = 4 - 5(x - 2)$, thus the direction (slope) is -5; and

(3) the equation of the best quadratic (parabolic) approximation to the graph of f is $y = 4 - 5(x - 2) - 5(x - 2)^2$, thus the graph lies below the tangent on both sides of the point under consideration.

Exercises 2-2

1. For each of the following express $f(x)$ in powers of the given factor.

   (a) $f(x) = 2x^3 - 5x$, $(x - 2)$

   (b) $f(x) = x^3 - 7x^2 + 3x + 4$, $(x - 2)$

   (c) $f(x) = 3x^3 - 5x^2 + 2x + 1$, $(x + 1)$

   (d) $f(x) = x^3 - 2x^2 + x - 1$, $(x + \frac{1}{2})$

2. For each of the following functions write the expansion of $f(x)$ in powers of $x - a$ and determine the equation of the tangent to the graph of f at the point $(a, f(a))$.

   (a) $f : x \rightarrow 3 + 4x + 2x^2 + x^3$, $a = 2$.

   (b) $f : x \rightarrow 3 + 2x^3 + 4x^2$, $a = -3$

   (c) $f : x \rightarrow 4x^3 - 3x^2 + 2x + 1$, $a = -4$

   (d) $f : x \rightarrow 5x^4 - 3x^2 + 2x + 1$, $a = \frac{1}{2}$

   (e) $f : x \rightarrow 4x^3 + x^2 + 3x$, $a = 3$

   (f) $f : x \rightarrow 2x^3 + x^2 - 16x - 24$, $a = -2$

3. For each of the following write the equation of the tangent at the specified point and sketch the shape of the graph nearby.

   (a) $x \rightarrow 4 + 3x - 7x^2 + x^3$ at $(2,-10)$

   (b) $x \rightarrow x^3 - 6x^2 + 6x - 1$ at $(3,-10)$

   (c) $x \rightarrow 3x^4 - 4x^3$ at $(1,-1)$

   (d) $t \rightarrow 2t^3 - 4t^2 - 5t + 9$ at $(2,-1)$

   (e) $x \rightarrow 2x^3 - 3x^2 - 12x + 14$ at $(1,1)$

   (f) $s \rightarrow 2s^3 - 6s^2 + 6s - 1$ at $(1,1)$

4. (a) $f(x) = x^3 - 3x$ in terms of ascending powers of $x - 2$.

   (b) Write inequalities to show the relative values of $(x - 2)$, $(x - 2)^2$, and $(x - 2)^3$ near the point where $x = 2$. (For instance consider $x = 1.9$ or $x = 2.1$.)

   (c) If $y = f(x) = x^3 - 3x$ write the value of y when $x = 2$.

   (d) Write the equation of the best linear approximation to the graph of $f : x \rightarrow y = x^3 - 3x$ near the point where $x = 2$.

(e) What is the direction (slope) of the (tangent to the) graph of $f : x \to x^3 - 3x$ near the point where $x = 2$?

(f) Write the equation of the best quadratic approximation to the graph of $f : x \to y = x^3 - 3x$ near the point where $x = 2$.

(g) What is the coefficient of x^2 in the parabola which best represents the graph of $f : x \to x^3 - 3x$ near the point where $x = 2$?

(h) Near the point where $x = 2$ is the graph of $f : x \to x^3 - 3x$ flexed (concave) upward or downward? Why?

(i) Compare the behavior of the graph of $f : x \to x^3 - 3x$ near the point where $x = 2$ with the behavior of the graph of $F : x \to 2 + 9x + 6x^2 + x^3$ at its y-intercept.

5. Again consider the function $f : x \to x^3 - 3x$.

(a) Beginning with the simple statement $x = a + (x - a)$, express x^3 and $-3x$ in terms of $x - a$. Write $x^3 - 3x$ in powers of $x - a$.

(b) Make a table to indicate three successive synthetic divisions of $x^3 - 3x$ and resulting quotients by $x - a$.

(c) Use your table from part (b) to write $x^3 - 3x$ in powers of $x - a$.

(d) Write inequalities to show the relationships between $(x - a)$, $(x - a)^2$, and $(x - a)^3$ when x in close to a.

(e) If $f : x \to x^3 - 3x$, find the value of f at a.

(f) What is the linear function that best approximates the graph of f at a?

(g) What is the direction (slope) of the (tangent to the) graph of f near the point where $x = a$?

(h) For what values of a does a tangent to the graph of f have zero slope?

(i) At what points is the tangent to the graph of f horizontal.

(j) What is the quadratic function that best approximates the graph of f near the point $(a,f(a))$?

(k) What is the coefficient of x^2 in the best parabolic representation to the graph of f near the point $(a,f(a))$?

(ℓ) For each of the points found in part (i) determine whether the (parabolic approximation to the) graph of f is flexed (concave) downward on upward.

(m) Decide which of the points found in part (i) is a relative maximum and which is a relative minimum.

(n) If the coefficient of the x^2 in the parabola which best represents the graph of f near some point $(a,f(a))$ is neither positive nor negative, then the graph is neither flexed upward nor downward at that point. (We refer to such a point as a point of inflection.) At what point on the graph of $f : x \rightarrow x^3 - 3x$ does this phenomenon occur?

(o) Use information acquired in other parts of this problem to quickly sketch the graph of $f : x \rightarrow x^3 - 3x$.

2-3. The Slope as Limit of Difference Quotients

To find the equation of the tangent line to the graph of a polynomial function f at the point $(a,f(a))$ we expressed f in terms of powers of $x - a$, and then omitted the terms of degree larger than 1. Thus, we wrote the function

$$f : x \rightarrow b_0 + b_1x + b_2x^2 + \dots + b_nx^n$$

as

$$f : x \rightarrow c_0 + c_1(x - a) + c_2(x - a)^2 + \dots + c_n(x - a)^n$$

to obtain the equation of the tangent line

$$y = c_0 + c_1(x - a)$$

to the graph of f at the point $(a,f(a))$.

We now describe an alternative procedure for finding the slope c_1 of this tangent line.

Let $P(a,f(a))$ be a point on the graph of f and let $Q(x,f(x))$ be a nearby point on the same graph. See Figure 2-3a where Q is to the right of P.

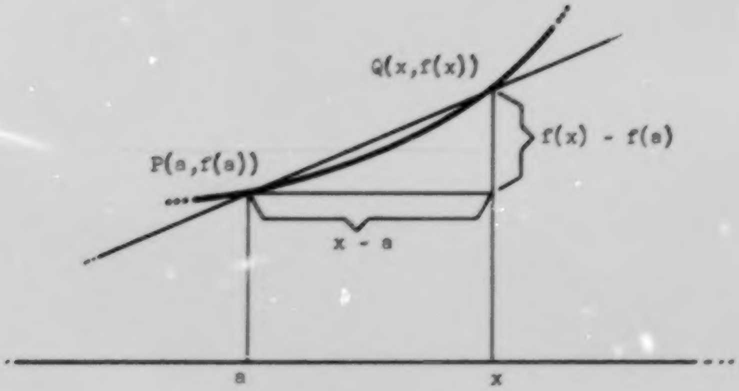


Figure 2-3a

The line that joins P and Q has the slope

$$\frac{f(x) - f(a)}{x - a} .$$

Consider what happens to this "difference quotient" if we choose Q on the graph closer and closer to P.

Figure 2-3b shows intuitively that the slope of the secant $\overleftrightarrow{PQ}$ is approaching the slope of the tangent $\overleftrightarrow{PT}$.

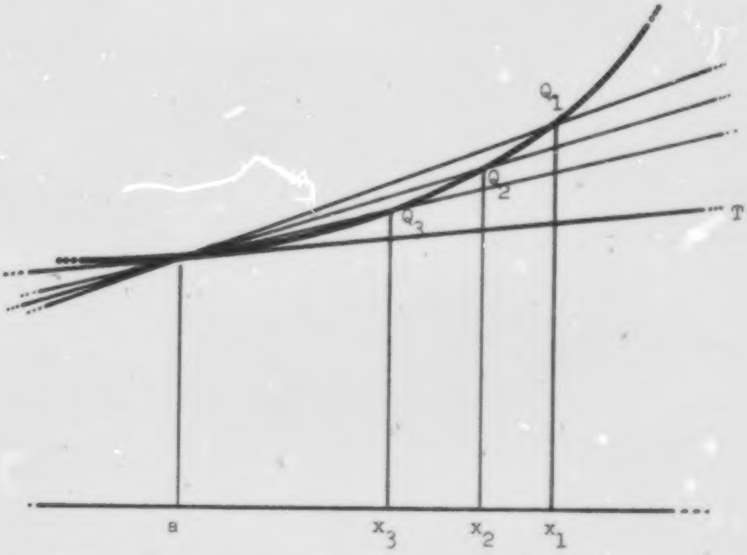


Figure 2-3b

To take a specific example, let

$$f : x \to 1 + x - 4x^2$$

and let P be the point $(0,1)$. Then

$$f(x) - f(a) = f(x) - f(0) = (1 + x - 4x^2) - 1 = x - 4x^2$$

and

$$x - a = x - 0 = x.$$

The slope of $\overleftrightarrow{PQ}$ is

$$\frac{f(x) - f(a)}{x - a} = \frac{x - 4x^2}{x} .$$

Since $x \neq 0$, we may divide and obtain the result

$$\text{slope}(PQ) = \frac{f(x) - f(a)}{x - a} = 1 - 4x.$$

If $x > 0$ the slope of $\overleftrightarrow{PQ}$ is less than 1, and if $x < 0$ the slope of $\overleftrightarrow{PQ}$ is greater than 1.

To take Q closer to P means to take x closer to zero, and hence to take $4x$ closer to zero, and $1 - 4x$ closer to 1. In fact, we can make $1 - 4x$ differ from 1 by as small amount as we please by choosing $|x|$ small enough.

In fact, if $x < \frac{\epsilon}{4}$, where ϵ is a positive number no matter how small, then

$$1 - 4x > 1 - 4(\tfrac{\epsilon}{4}) = 1 - \epsilon.$$

Similarly, if $x > -\frac{\epsilon}{4}$, then

$$1 - 4x < 1 - 4(-\tfrac{\epsilon}{4}) = 1 + \epsilon.$$

Since the slope of $\overleftrightarrow{PQ}$ is $1 - 4x$ we conclude

$$1 - \epsilon < \text{slope}(PQ) < 1 + \epsilon$$

provided that $-\frac{\epsilon}{4} < x < \frac{\epsilon}{4}$; that is, if $|x| < \frac{\epsilon}{4}$.

For smaller and smaller choices of ϵ (> 0) the slope of the secant $\overleftrightarrow{PQ}$ is thus brought arbitrarily close to 1. We have learned to describe this by saying that slope of $\overleftrightarrow{PQ}$ approaches 1 as x approaches 0. In this example we shall call the number 1 the limit of slope of $\overleftrightarrow{PQ}$ as Q approaches P, or as x approaches 0. In Figure 2-3b the line $\overleftrightarrow{PT}$ with this limiting slope is the tangent to the graph at the point P.

Generalizing from this example, we introduce a definition.

The slope of the tangent to the graph of f at the point $P(a,f(a))$ is the limit of

$$\frac{f(x) - f(a)}{x - a}$$

as x approaches a.

A convenient abbreviation for this phrase is

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

to be read "the limit, as x approaches a, of $f(x) - f(a)$ divided by $x - a$."

We illustrate this definition by using the same function

$$f : x \to 1 + x - 4x^2$$

but a different point $P(1,-2)$ on its graph. Let Q be the point $(x,f(x))$.

Then

$$\begin{aligned} \text{slope}(PQ) &= \frac{f(x) - f(1)}{x - 1} \\ &= \frac{(1 + x - 4x^2) - (-2)}{x - 1} \\ &= \frac{3 + x - 4x^2}{x - 1} \\ &= -\frac{4x^2 - x - 3}{x - 1} \\ &= -\frac{(x - 1)(4x + 3)}{x - 1} \\ &= -(4x + 3) \qquad [\text{since } x \neq 1] \end{aligned}$$

Now $\lim_{x \to 1} -(4x + 3) = -7$ which is the slope of the tangent to the graph at $(1,-2)$.

We illustrate the use of our definition with two further examples.

Example 2-5a. Find the slope of the tangent to the graph of

$$f : x \to 2x - x^3$$

at the point $P(a,f(a))$. The slope of the line through $P(a,f(a))$ and $Q(x,f(x))$, $x \neq a$, is given by the difference quotient

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} &= \frac{(2x - x^3) - (2a - a^3)}{x - a} \\ &= \frac{2(x - a) - (x^3 - a^3)}{x - a} \\ &= 2\left(\frac{x - a}{x - a}\right) - \left(\frac{x^3 - a^3}{x - a}\right) \\ &= 2 - (x^2 + ax + a^2), \quad x \neq a. \end{aligned}$$

As x approaches a the difference quotient approaches

$$2 - (a^2 + aa + a^2) = 2 - 3a^2.$$

We conclude that

$$2 - 3a^2 = \text{slope of tangent to the graph of } f \text{ at } P(a,f(a)).$$

Example 2-5b. Find the slope of the tangent to

$$f : x \to 1 - 2x + x^2 - 3x^4$$

at the point $P(a,f(a))$.

The desired slope will be the limit of the difference quotient

$$\frac{f(x) - f(a)}{x - a}, \text{ as } x \text{ approaches } a.$$

Using the expression for f we have

$$\begin{aligned}\frac{f(x) - f(a)}{x - a} &= \frac{(1 - 2x + x^2 - 3x^4) - (1 - 2a + a^2 - 3a^4)}{x - a}\\ &= 2\left(\frac{x - a}{x - a}\right) + \frac{x^2 - a^2}{x - a} - 3\left(\frac{x^4 - a^4}{x - a}\right)\\ &= 2 + (x + a) - 3(x^3 + ax^2 + a^2x + a^3), \quad x \neq a.\end{aligned}$$

As x approaches a,

$$x + a \text{ approaches } 2a$$

and

$$-3(x^3 + ax^2 + a^2x + a^3) \text{ approaches } -3(a^3 + aa^2 + a^2a + a^3) = -12a^3$$

so that

$$\frac{f(x) - f(a)}{x - a} \text{ approaches } 2 + 2a - 12a^3, \text{ the desired slope.}$$

Exercises 2-3

1. For each of the following functions, assume that $x \neq a$ and write the difference quotient

$$r(x) = \frac{f(x) - f(a)}{x - a}$$

in simplest form.

(a) $f : x \to x$

(b) $f : x \to x^2$

(c) $f : x \to x^3$

(d) $f : x \to x^4$

2. For each of the functions in Number 1 evaluate the limit as x approaches a of the difference quotient $r(x)$.

3. Find the slope of the tangent to the graph of each of the functions in Number 1 at the point $(a,f(a))$.

4. Find the slope of the tangent to the graph of each of the functions in Number 1 at the point $(0,0)$.

5. Write the equation of the tangent line to the graph of each of the functions in Number 1 at the point $(a,f(a))$.

6. For each of the following functions, assume that $x \neq a$ and write the difference quotient

$$r(x) = \frac{f(x) - f(a)}{x - a}$$

in simplest form.

(a) $f : x \to mx + b$

(b) $f : x \to Ax^2 + Bx + C$

(c) $f : x \to Ax^3 + Bx^2 + Cx + D$

7. For each of the functions in Number 6 evaluate the limit as x approaches a of the difference quotient $r(x)$.

8. Find the slope of the tangent to the graph of each function in Number 6 at the point $(a,f(a))$.

9. Find the slope of the tangent to the graph of $f : x \to 20x - 3x^2$ at the point $(a,f(a))$.

10. Consider the function $f : x \to 1 - x^3$.

(a) Evaluate $\lim\limits_{z \to x} \frac{f(z) - f(x)}{z - x}$; i.e., determine the limit as z approaches x, of the difference quotient $\frac{f(z) - f(x)}{z - x}$.

(b) Evaluate $\lim\limits_{h \to 0} \frac{f(x + h) - f(x)}{h}$.

(c) What is the slope of the tangent to the graph of f at the point $(x,f(x))$?

11. Consider the function

$$f : x \to 1 + x - 4x^2.$$

(a) Find the limit as x approaches a of the difference quotient

$$\frac{f(x) - f(a)}{x - a}.$$

(b) What is the slope of the tangent to the graph of f at the point $(a,f(a))$?

(c) Find the limit as h approaches zero of the difference quotient

$$\frac{f(x + h) - f(x)}{h}.$$

(d) What is the slope of the tangent to the graph of f at the point $(x,f(x))$?

2-4. The Derivative

Consider the function

$$f : x \to 1 + x - 4x^2. \tag{1}$$

In the previous section we showed that the tangent line to the graph of f at $(a,f(a))$ has slope $1 - 8a$. The slope of the tangent to the graph of f at the point $(x,f(x))$ is given by $1 - 8x$. (See No. 10(c), Exercises 2-3.)

The function

$$x \to 1 - 8x \tag{2}$$

is sometimes called the *slope function* for $f : x \to 1 + x - 4x^2$ since its value $1 - 8x$ at a point $(x,f(x))$ gives the slope of the tangent line to the graph of f at the point $(x,f(x))$. The function (2) is more commonly known as the *derivative*[*] of f and usually denoted by f'. Thus the derivative of

$$f : x \to 1 + x - 4x^2$$

is the function

$$f' : x \to 1 - 8x.$$

The value $f'(x)$ at a point $(x,f(x))$ is the slope of the tangent to the graph of f at $(x,f(x))$. For brevity, we will often refer to this value $f'(x)$ as simply the *slope of the graph* of f at $(x,f(x))$.

Our purpose in this section and the next is to obtain a formula for the derivative f' (that is, the slope function) of an arbitrary polynomial function f. In the previous section we defined the slope of (the tangent to) the graph of f at the point $(a,f(a))$ to be

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{3}$$

An alternate form will be more convenient here. If we write $(x + h)$ in place of x in (3), and substitute x in place of a, (3) becomes

$$f'(x) = \lim_{(x+h) \to x} \frac{f(x + h) - f(x)}{(x + h) - x},$$

which simplifies to

---

[*] The name is reserved for this very special function, in spite of the fact that there are many functions which could be derived in other ways from a particular function under consideration.

$$(4) \qquad f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} .$$

Note that $\frac{f(x + h) - f(x)}{h}$ can still be interpreted as the slope of a secant $\overleftrightarrow{PQ}$, where P has coordinates $(x,f(x))$ and Q has coordinates $(x + h , f(x + h))$. (See Figure 2-4a.)

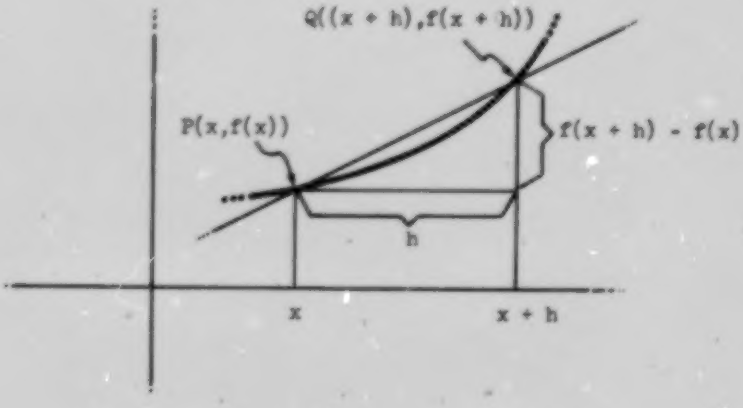


Figure 2-4a

It is clear from Figure 2-4a that the slope of $\overleftrightarrow{PQ}$ is $\frac{f(x + h) - f(x)}{h}$. As in the last section we define the slope of the tangent to be the limit of this difference quotient as Q approaches P, that is, in terms of the new expression (4), as h approaches zero. We are denoting this limit by $f'(x)$ and calling f' the derivative of f.

Example 2-4a. Given the function $f : x \to 3x^2 - 2x + 1$, use (4) to find $f'(x)$ and the slope of (the tangent to) the graph of f at the point $(2,9)$.

$$\begin{aligned}
f(x + h) &= 3(x + h)^2 - 2(x + h) + 1 \\
&= 3x^2 + 6xh + 3h^2 - 2x - 2h + 1 \\
f(x) &= 3x^2 - 2x + 1 \\
f(x + h) - f(x) &= 6xh + 3h^2 - 2h \\
\frac{f(x + h) - f(x)}{h} &= 6x + 3h - 2 \\
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} &= 6x - 2 = f'(x).
\end{aligned}$$

The slope of the tangent at $(2,9)$ is $f'(2) = 6 \cdot 2 - 2 = 12 - 2 = 10$.

Example 2-4b. To find the equation of the tangent to the graph of

$$f : x \to x^3$$

at (1,1).

$$f'(x) = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}$$

$$(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3.$$

Hence

$$(x + h)^3 - x^3 = 3x^2h + 3xh^2 + h^3,$$

$$\frac{(x + h)^3 - x^3}{h} = 3x^2 + 3xh + h^2,$$

and

$$f'(x) = 3x^2.$$

The required tangent has the equation

$$\begin{aligned} y &= f(1) + f'(1)(x - 1) \\ &= 1 + 3(x - 1) \end{aligned}$$

or

$$y = 3x - 2.$$

This solution has the advantage that it enables us to obtain tangents at other points with little extra work. Thus at (2,8), the tangent has the slope $f'(2) = 3 \cdot 2^2 = 12$.

Exercises 2-4

1. Consider the function $f : x \to x^2 - 1$.

   (a) Find $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f'(x)$.

   (b) Find $f'(3)$ from part (a).

   (c) Find $f'(3)$ as $\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$.

   (d) Construct a table of values for $\frac{f(3 + h) - f(3)}{h}$ using successively $h = .1, .01, .001$, and also $h = -.1, -.01, -.001$.

2. Use the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

   to find $f'(x)$ for each of the following:

   (a) $f(x) = x^2 - x + 1$

   (b) $f(x) = 3x^2 + 4$

   (c) $f(x) = 2x^2 - x + 4$

3. If $f : x \to ax^2 + bx + c$ where a, b, and c are constants, show that $f' : x \to 2ax + b$.

4. Use the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

   to find the derivative of each of the following:

   (a) $f(x) = x^3 + x$

   (b) $f(x) = x^3 - 3x$

   (c) $f(x) = 2x^3 + x^2 - 6x + 3$

5. If $f(x) = ax^3 + bx^2 + cx + d$, show that $f'(x) = 3ax^2 + 2bx + c$.

6. If $f : x \to 1 + 2x - x^2$ evaluate

   (a) $f'(a)$

   (b) $f'(0)$

   (c) $f'(\frac{1}{2})$

   (d) $f'(1)$

   (e) $f'(-10)$

7. If $f : x \to 1 + 2x - x^2$, find the slope of the tangent to the graph of f at each of the following points.

   (a) $(a, f(a))$

   (b) $(0, f(0))$

   (c) $(\frac{1}{2}, f(\frac{1}{2}))$

   (d) $(1, f(1))$

   (e) $(-10, f(-10))$

8. If $f : x \to x^3 - 2x + 1$, find all x such that

   (a) $f'(x) = 4$

   (b) $f'(x) = 22$

   (c) $f'(x) = 0$

   (d) $f'(x) = -1$

9. Determine each of the following.

   (a) $\lim\limits_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$

   (The symbol "Δx," read "delta x," often stands for "change in x." It is merely another name for the quantity h in (4).)

   (b) $\lim\limits_{x \to a} \frac{x^4 - a^4}{x - a}$

   (c) $\lim\limits_{z \to x} \frac{z^5 - x^5}{z - x}$

   (d) $\lim\limits_{h \to 0} \frac{(x + h)^6 - x^6}{h}$

10. (a) What is the slope function of $f : x \to x^3$?

    (b) What is the derivative of $f : x \to x^4$?

    (c) What is the slope of (the tangent to) the graph of $f : x \to x^5$ at the point $(x, f(x))$?

    (d) What is $f'(x)$ if $f : x \to x^6$?

11. Find f' if

    (a) $f : x \to (x + 1)^2$

    (b) $f : x \to x(3x + 1)^2$

    (c) $f : x \to (x^2 + 2x)(3x - 1)$

12. The graph of $f : x \to Ax^2 + Bx + C$ includes the point $(1,8)$, and the slope of the graph of f at $(a,f(a))$ is $6a - 2$. Determine A, B, and C.

2-5. Derivatives of General Polynomial Functions

In Exercises 3 and 5, Section 2-4, you were asked to show that if

$$f : x \to ax^2 + bx + c$$

then

$$f' : x \to 2ax + b,$$

and if

$$g : x \to ax^3 + bx^2 + cx + d$$

then

$$g' : x \to 3ax^2 + 2bx + c$$

Such general expressions allow us to write derivatives of specific quadratic and cubic functions by inspection. For example, to obtain the derivative of the function

$$g : x \to 5x^3 - 7x^2 - 11x + 13$$

we merely observe that this is the above cubic polynomial with $a = 5$, $b = -7$, $c = -11$ and $d = 13$, so we know immediately that

$$g'(x) = 3(5x^2) + 2(-7x) + (-11) = 15x^2 - 14x - 11.$$

These examples, and other exercises in Section 2-4, suggest the following general expressions for derivatives of polynomial functions:

(1)

The monomial

$$f : x \to bx^n$$

has the derivative

$$f' : x \to nbx^{n-1}$$

(2)

The polynomial

$$f : x \to b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-1} + b_nx^n$$

has the derivative

$$f' : x \to b_1 + 2b_2x + 3b_3x^2 + \dots + (n-1)b_{n-1}x^{n-2} + nb_nx^{n-1}$$

Because derivatives are calculated as limits of difference quotients, the process of obtaining the derivative of a function is called differentiation. Observe that (2) states that the derivative of a general n^{th} degree polynomial can be obtained by differentiating each term in the sum, according to (1). We are claiming that the derivative of such a sum of terms is simply the sum of the derivatives of the terms.

Formulas (1) and (2) can both be derived by writing the expansions for $\frac{f(x+h)-f(x)}{h}$ using the Binomial Theorem, and taking the limit as h approaches zero. We shall derive (1) this way. Fortunately, however, we can deduce (2) from (1) without a long algebraic argument by first justifying the claim we made above, namely, that the derivative of a sum of functions is the sum of their derivatives.

To show (1), let $f : x \to bx^n$. Then according to the Binomial Theorem,

$$f(x+h) = b(x+h)^n$$
$$= b[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{2 \cdot 3}x^{n-3}h^3 + \ldots + h^n].$$
$$\therefore \quad f(x+h) - f(x) = b[nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n]$$

and

$$(3) \qquad \frac{f(x+h)-f(x)}{h} = b[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1}].$$

Note that every term in (3) except the first term, bnx^{n-1}, contains h at least once as a factor. Hence, as h approaches zero, all the terms in (3) except bnx^{n-1} also approach zero. We conclude that

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = bnx^{n-1}$$

and therefore, that the derivative of $f : x \to bx^n$ is $f' : x \to bnx^{n-1}$.

Now to see that (2) follows from this result we must first see why the general property of derivatives mentioned above holds true. Suppose s is a function whose value $s(x)$ equals $f(x) + g(x)$, where f and g are functions of x. (For example, if $f : x \to 5x^4$ and $g : x \to 13x^7$, then $s : x \to 5x^4 + 13x^7$.) We may calculate the derivative of s in general directly from the definition, as follows:

$$s'(x) = \lim_{h \to 0} \frac{s(x+h)-s(x)}{h}$$

$$s(x+h) = f(x+h) + g(x+h)$$

and $$s(x) = f(x) + g(x)$$

so

$$s(x + h) - s(x) = f(x + h) - f(x) + g(x + h) - g(x)$$

Therefore

$$\frac{s(x + h) - s(x)}{h} = \frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h}. \tag{4}$$

In the limit as h approaches zero, it can be shown that (4) becomes

$$s'(x) = f'(x) + g'(x),$$

as claimed.

The argument could easily be extended to sums of three, four, or any number of functions. Hence, we see that we can differentiate the general polynomial function

$$f : x \to b_0 + b_1x + b_2x^2 + \dots + b_nx^n$$

term by term since it can be thought of as the sum of $(n + 1)$ functions. From (1) we conclude that

$$f'(x) = b_1 + 2b_2x + 3b_3x^2 + \dots + nb_nx^{n-1}.$$

Translation and the Derivative

Now let us consider what happens to our differentiation formulas if we replace x by $x - a$. To be concrete we consider the function

$$f : x \to 2x^2 - 8x + 9,$$

whose derivative is

$$f' : x \to 4x - 8.$$

Let g be the function

$$g : x \to 2(x - a)^2 - 8(x - a) + 9,$$

that is, $g(x) = f(x - a)$. If $a = -6$, the graph of g is obtained by translating the graph of f six units to the left. (See Figure 2-5a.)

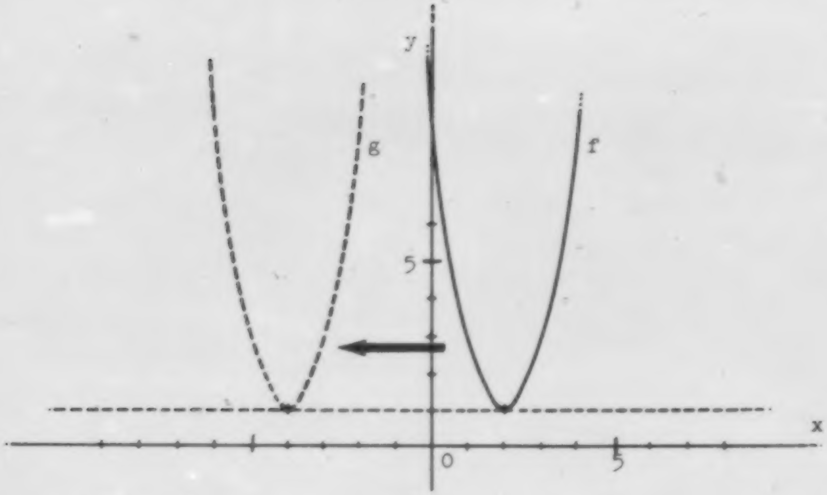


Figure 2-5a

$f : x \rightarrow 2x^2 - 8x + 9$

$g : x \rightarrow (2x + 6)^2 - 8(x + 6) + 9$

If $a = 7$, g becomes $g : x \rightarrow 2(x - 7)^2 - 8(x - 7) + 9$ and its graph is obtained by translating the graph of f seven units to the right. (See Figure 2-5b.)

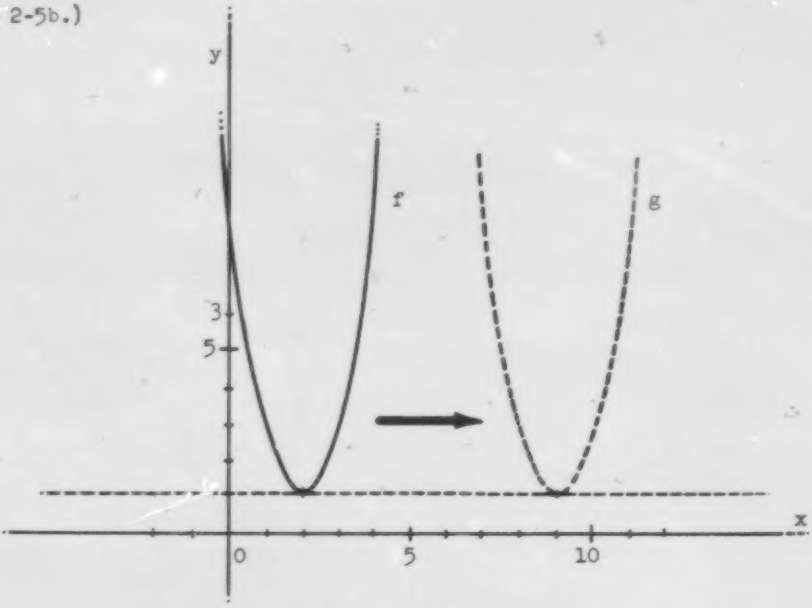


Figure 2-5b

$f : x \rightarrow 2x^2 - 8x + 9$

$g : x \rightarrow 2(x - 7)^2 - 8(x - 7) + 9.$

Now under a translation a line is carried into a parallel line, and hence, the slope of a tangent to a graph is unaffected by translation. (See Section 1-2.) Thus, if $g : x \to f(x - a)$, that is, the graph of g is the result of translating the graph of f, then the slope of the graph of g at the point $(x,g(x))$ is the same as the slope of the graph of f at the corresponding point $(x - a , f(x - a))$. For example, the slope of the graph of $g : x \to 2(x - 7)^2 - 8(x - 7) + 9$ at $(10,3)$ is exactly the same as the slope of the graph of $f : x \to 2x^2 - 8x + 9$ at $(3,3)$, as shown in Figure 2-5c, since T' is the image of T under a translation 7 units to the right.

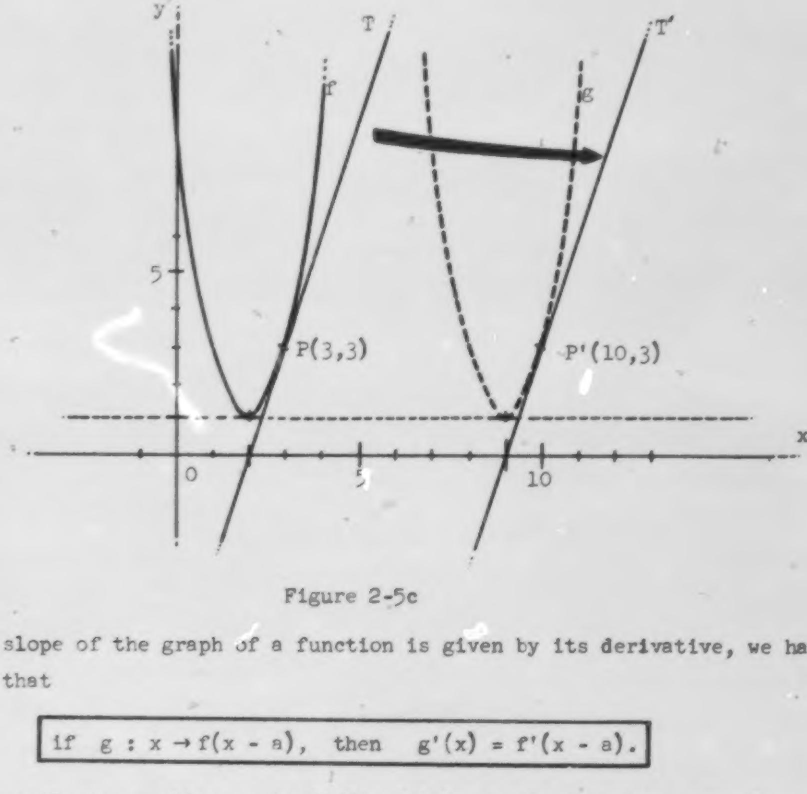


Figure 2-5c

Since the slope of the graph of a function is given by its derivative, we have concluded that

$$(5) \qquad \boxed{\text{if } \ g : x \to f(x - a), \ \text{ then } \ g'(x) = f'(x - a).}$$

This conclusion enables us to differentiate g in the above example rather easily. For the function

$$f : x \to 2x^2 - 8x + 9$$

we find the derivative

$$f' : x \to 4x - 8$$

and replace x by $x - a$ to obtain

$$g' : x \to 4(x - a) - 8.$$

For example, if $a = -6$, we have

$$g : x \rightarrow 2(x + 6)^2 - 8(x + 6) + 9$$

and

$$g' : x \rightarrow 4(x + 6) - 8 = 4x + 16.$$

We can also justify conclusion (5) algebraically using difference quotients.

We know that

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \tag{6}$$

Replacing x by $x - a$ in (6) we obtain

$$f'(x - a) = \lim_{h \to 0} \frac{f(x - a + h) - f(x - a)}{h}.$$

Now if $g(x) = f(x - a)$ we can rewrite this as

$$f'(x - a) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h},$$

and the right-hand expression is simply the definition of $g'(x)$. Hence,

$$f'(x - a) = g'(x).$$

In general then,

(7)

The polynomial function

$$g : x \rightarrow c_0 + c_1(x - a) + c_2(x - a)^2 + \ldots + c_n(x - a)^n$$

has the derivative

$$g' : x \rightarrow c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \ldots + nc_n(x - a)^{n-1}.$$

Example 2-5a. Given the function $f : x \rightarrow 3x^2 - 2x + 1$, use (2) to find the derivative f' and the slope of the graph of f at the point $(2,9)$.

Using (2), we obtain $f' : x \rightarrow (2)(3x) - (1)(2) = 6x - 2$; $f'(2) = 10$. The slope of the graph of f at $(2,9)$ is 10.

Example 2-5b. Given $f : x \to x^5 - 3x^2 + x - 6$, find the equation of the tangent line to the graph of f at the point where $x = 1$.

Since $f(1) = -7$, the tangent passes through $(1,-7)$. The derivative of f is

$$f' : x \to 5x^4 - 6x + 1$$

so that the slope of the tangent at $(1,-7)$ is $f'(1) = 0$. Thus, the tangent line at $(1,-7)$ is horizontal and has the equation

$$y = -7.$$

Note how much easier it is to find the equation of this tangent by using the derivative formula (2) to obtain its slope at $x = 2$, rather than using the method of expressing $f(x)$ in powers of $x - 1$ as we did in Section 2-2.

Example 2-5c. Find the equation of the tangent to the graph of $y = -4x^3 - 7x + 1$ at the point $(2,-45)$.

It is common to denote the expression for the derivative by y', so that (2) gives

$$y' = -12x^2 - 7.$$

This is the slope of the tangent to the graph at any point (x,y). To find the slope of the tangent at the point $(2,-45)$ we replace x by 2 to obtain

$$-12 \cdot 2^2 - 7 = -55.$$

The equation of the tangent line is

$$y = -45 - 55(x - 2).$$

Example 2-5d. For $f : x \to x^3 - 3x^2$, and $g : x \to f(x - \frac{5}{2})$, find $g'(1)$.

We have

$$f' : x \to 3x^2 - 6x$$

so that

$$g' : x \to 3(x - \tfrac{5}{2})^2 - 6(x - \tfrac{5}{2})$$

and hence

$$g'(1) = 3(1 - \frac{5}{2})^2 - 6(1 - \frac{5}{2}) = \frac{63}{4} .$$

Example 2-5e. Suppose $f : x \rightarrow (x + 1)^{10}$. Find the equation of the tangent to the graph of f at the point $(1, 1024)$.

We could use the binomial theorem to expand $(x + 1)^{10}$ and then differentiate. However, it is easier to use (7) which gives

$$f' : x \rightarrow 10(x + 1)^9$$

so that

$$f'(1) = 10 \times 2^9 = 5120.$$

Hence, the desired tangent has the equation

$$y = 1024 + 5120(x - 1).$$

Exercises 2-5

1. (a) Find f' if $f : x \rightarrow x^2 + 2x + 1$ using (2).

   (b) Find g' if $g : x \rightarrow (x + 1)^2$ using (7).

   (c) Compare f' and g'.

2. (a) Find the derivative of each of the following functions.

   (i) $f : x \rightarrow 2x^2 - 8x + 9$

   (ii) $g_1 : x \rightarrow 2(x + 5)^2 - 8(x + 6) + 9$

   (iii) $g_2 : x \rightarrow 2(x - 7)^2 - 8(x - 7) + 9$

   (b) Find the slope of the tangent to the graph of each of the functions in part (a) at the point indicated:

   (i) f at $(3, f(3))$;
   (ii) g_1 at $(-3, g_1(-3))$;
   (iii) g_2 at $(10, g_2(10))$.

   (c) Show that the tangents to the graphs of f, g_1, and g_2 at the points indicated in part (b) are parallel lines.

   (d) Indicate the function obtained by shifting the graph of each of the functions in part (a) as prescribed:

   (i) the graph of f two units to the left;
   (ii) the graph of g_1 four units to the right;
   (iii) the graph of g_2 nine units to the left.

3. (a) Find the derivatives of each of the following functions.

   (i) $F : x \rightarrow x^3 - 3x$

   (ii) $f : x \rightarrow (x - 2)^3 + 6(x - 2)^2 + 9(x - 2) + 2$

   (iii) $g : x \rightarrow (x + 1)^3 - 3(x + 1)^2 + 2$

   (b) Evaluate:

   (i) $F'(0)$
   (ii) $f'(0)$
   (iii) $g'(0)$

   (c) What is the equation of the tangent to the graph of each of the functions F, f, and g at the y-axis?

(d) Compare the functions F, f, and g.

4. (a) Find the derivative of

$$F : x \to x^3 + 6x^2 + 12x + 8 .$$

(b) Determine f' if $f : x \to (x + 2)^3$.

(c) Evaluate $\lim\limits_{h \to 0} \dfrac{(x + 2 + h)^3 - (x + 2)^3}{h}$.

(d) Evaluate $F'(-1)$.

(e) Evaluate $f'(-1)$.

(f) Evaluate $\lim\limits_{x \to -1} \dfrac{(x + 2)^3 - (-1 + 2)^3}{x + 1}$

5. Consider the function $f : x \to (x + 1)^{10}$.

(a) Find f'.

(b) Evaluate $f(0)$ and $f'(0)$.

(c) What is the equation of the tangent to the graph of f at the y-axis?

(d) Evaluate $f(-1)$ and $f'(-1)$.

(e) Find the equation of the tangent to the graph of f at the point where $x = -1$.

(f) Evaluate $f(-2)$ and $f'(-2)$.

(g) Write the equation of the tangent to the graph of f at the point where $x = -2$.

6. Consider the function $f : x \to (x - 2)^{15}$.

(a) Find the derivative of f.

(b) Evaluate $f'(1)$, $f'(2)$, and $f'(3)$.

(c) Find the equation of the tangent to the graph of f at the point $(4 , 32768)$.

7. (a) Find f' if $f : x \to 3(x + 2)^2$.

(b) What is the derivative of $g : x \to 3x^2 + 12x + 12$?

(c) Compare f with g, and f' with g'.

(d) For $F : x \to (3x + 6)^2$ find F'.

(e) Determine the derivative of $G : x \to 9(x + 2)^2$.

(f) Compare F with G, and F' with G'.

8. Find f' if

(a) $f : x \to 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

(b) $f : x \to x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!}$

(c) $f : x \to 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!}$

(Note: "n!", read "n factorial," is the product of all the integers from 1 through the positive integer n.

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n - 1) \cdot n.$$

For example, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.)

9. (a) Find two points where the slope of the graph of $f : x \to 2x^3 - 9x^2 - 60x + 5$ is zero.

(b) What does the graph of f look like at these two points?

10. Consider the functions

$$f : x \to x^3 - 3x^2 + 1 \quad \text{and} \quad g : x \to \frac{x^2}{2} - \frac{2}{3}x - \frac{5}{6}.$$

(a) Find the associated slope functions f' and g'.

(b) Evaluate $f'(1)$ and $g'(1)$.

(c) In each case write an equation of the line tangent to the graph of the function at the point where $x = 1$.

(d) What is the relationship of these tangent lines to one another?

11. (a) Using $x = (x - a) + a$ and the Binomial Theorem express x^7 in powers of $x - a$.

(b) Using part (a) determine $\frac{x^7 - a^7}{x - a}$ for $x \neq a$.

(c) Evaluate $\lim_{x \to a} \frac{x^7 - a^7}{x - a}$.

(d) Determine $\frac{(x + \Delta x)^7 - x^7}{\Delta x}$ for $\Delta x \neq 0$. (See No. 9, Exercises 2-4.)

(e) As x increases by an amount Δx, the change in y is $(x + \Delta x)^7 - x^7$. This quantity is often labeled "Δy," representing "change in y." Determine

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} .$$

(This limit is, of course, the derivative of $y = x^7$, and is often symbolized by y' or $\frac{dy}{dx}$.)

2-6. Applications of the Derivative to Graphing

The derivative f' of a polynomial function f is very useful for obtaining information about the graph of f. In particular, the sign of $f'(x)$ will enable us to determine exactly the intervals over which the graph of f is rising or falling and to locate precisely the high and low points of the graph.

To be specific, consider the function

$$f : x \to 2x^3 - 3x^2 - 12x + 2.$$

Its derivative is given by

$$f' : x \to 6x^2 - 6x - 12.$$

The value $f'(x)$ can be interpreted as the slope of the graph of f at the point $(x, f(x))$. In factored form

$$f'(x) = 6(x + 1)(x - 2),$$

from which it follows that

$$\begin{aligned}
&f'(x) > 0 \quad \text{for} \quad x < -1,\\
&f'(x) = 0 \quad \text{for} \quad x = -1,\\
&f'(x) < 0 \quad \text{for} \quad -1 < x < 2,\\
&f'(x) = 0 \quad \text{for} \quad x = 2,\\
\text{and} \qquad &f'(x) > 0 \quad \text{for} \quad x > 2.
\end{aligned}$$

See Figure 2-6a for the graph of $f : x \to 2x^3 - 3x^2 - 12x + 2$, together with these facts.

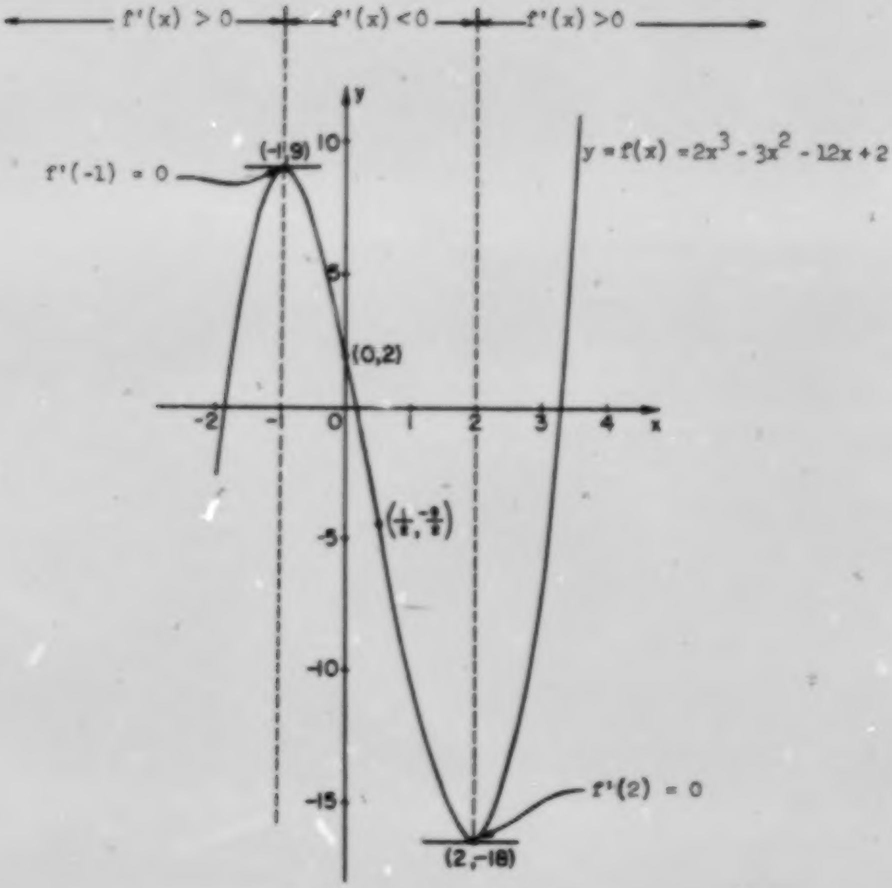


Figure 2-6a

The graph of $f : x \to 2x^3 - 3x^2 - 12x + 2$.

Note that as x increases, the graph of f rises over the intervals in which $f'(x) > 0$ and falls over the intervals in which $f'(x) < 0$. This is as we might expect from our experiences with positive and negative slopes of lines.

We now show, for any function f, that if $f'(a) > 0$, the graph of f is rising as x increases through a.

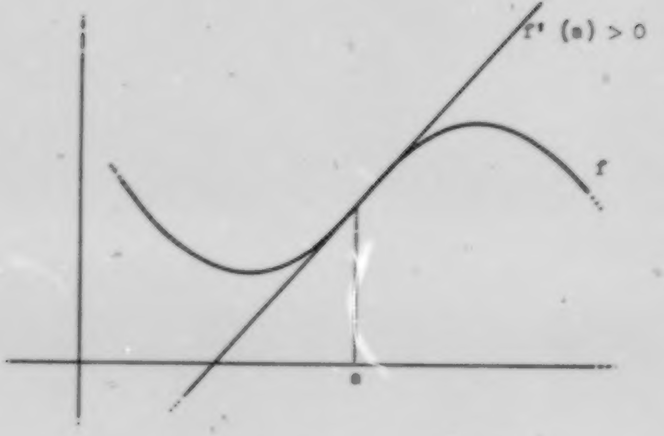


Figure 2-6b

By definition,

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Hence, if $f'(a) > 0$, then

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} > 0.$$

This limit can not be greater than zero unless

$$(1) \qquad \frac{f(a + h) - f(a)}{h} > 0$$

for sufficiently small values of h. Suppose we take h positive and small enough so that (1) holds. Then multiplying by h in (1) we have

$$f(a + h) - f(a) > 0$$

or

$$(2) \qquad f(a + h) > f(a).$$

This inequality (2) says simply that the graph of f rises immediately to the right of a. (See Figure 2-6c.)

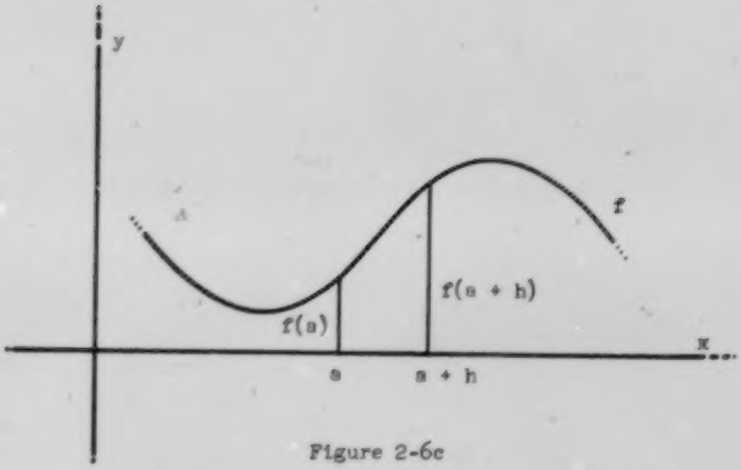


Figure 2-6c

$f(a + h) > f(a)$ for h sufficiently small and positive.

Similarly, if we take h negative in (1), then on multiplication by h, (1) becomes

$$f(a + h) - f(a) < 0$$

or

$$f(a + h) < f(a),$$

which says that the graph of f falls away immediately to the left of a. (See Figure 2-6d.)

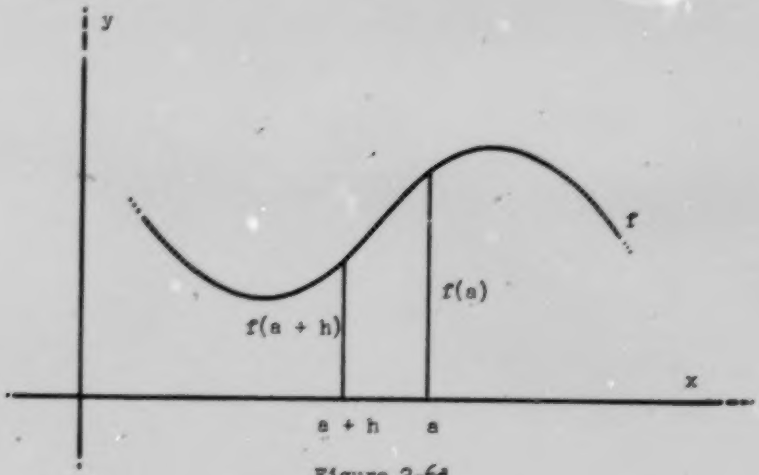


Figure 2-6d

$f(a + h) < f(a)$ for h sufficiently small and negative.

We have shown the following.

(3) If $f'(a) > 0$, then $f(x)$ increases at the point $(a,f(a))$, and the graph of f rises as x increases through the value $x = a$.

A completely analogous argument may be carried out for the case in which $f'(a) < 0$. We merely state the result.

(4) If $f'(a) < 0$, then $f(x)$ decreases at the point $(a,f(a))$, and the graph of f falls as x increases through the value $x = a$.

A simple but important corollary of (3) and (4) is the following.

(5) The graph of f is horizontal at the point $(a,f(a))$ only if $f'(a) = 0$.

For if the graph is horizontal at $(a,f(a))$, then $f(x)$ is neither increasing nor decreasing at $x = a$, so $f'(a)$ can neither be greater than zero, according to (3), nor less than zero, according to (4). Hence, $f'(a)$ must be equal to zero.

Returning to the function $f : x \rightarrow 2x^3 - 3x^2 - 12x + 2$ we note that we could have predicted its intervals of increase and decrease without having seen the graph in Figure 2-6a. Furthermore, we could have pinpointed exactly the location of the point $(-1,9)$ where the graph of this particular function ceases to rise and begins to fall (as x is increasing). Such a point is called a relative maximum of the function f because the value of f at that point is greater than all the values of f nearby. The point $(2,-18)$ is similarly called a relative minimum.

The point $(a,f(a))$ is a relative maximum of f if and only if in some interval about a the graph of f rises for $x < a$ and falls for $x > a$. In other words,

the point $(a,f(a))$ is a relative maximum if and only if

(i) $f'(a) = 0$
(ii) $f'(x) > 0$ for $x < a$ and close to a
(iii) $f'(x) < 0$ for $x > a$ and close to a.

The point $(a,f(a))$ is a relative minimum if and only if in some interval about a the graph of f falls for $x < a$ and rises for $x > a$. In other words,

the point $(a,f(a))$ is a relative minimum if and only if

(i) $f'(a) = 0$
(ii) $f'(x) < 0$ for $x < a$ and close to a
(iii) $f'(x) > 0$ for $x > a$ and close to a.

A final word about notation: in discussing the intervals over which a function is increasing or decreasing, it is convenient to use the symbol $[a,b]$ to represent the closed interval from a to b, including the endpoints a and b. That is, $[a,b]$ is the set of all x such that $a \leq x \leq b$. Often it is necessary to distinguish $[a,b]$ from the open interval, denoted (a,b), which excludes the endpoints a and b. That is, the interval (a,b) is the set of all x such that $a < x < b$. Note, for example, that the derivative f' of $f : x \to 2x^3 - 3x^2 - 12x + 2$ in Figure 2-6a is strictly less than zero on the open interval $(-1,2)$ while in the closed interval $[-1,2]$ this is not so.

Example 2-6a. Determine the relative maximum and minimum as well as intervals of increase and decrease for $f : x \to 1 + x - x^2 - x^3$.

We have

$$f' : x \to 1 - 2x - 3x^2 = -(3x - 1)(x + 1).$$

The graph of f has a horizontal tangent when $x = \frac{1}{3}$ and when $x = -1$. Examination of the signs of these factors leads to the conclusions:

$$f'(x) < 0 \quad \text{if} \quad x < -1$$
$$f'(x) > 0 \quad \text{if} \quad -1 < x < \frac{1}{3}$$
$$f'(x) < 0 \quad \text{if} \quad x > \frac{1}{3} .$$

Thus, f decreases if $x < -1$ or if $x > \frac{1}{3}$ and increases when x is between -1 and $\frac{1}{3}$. In particular

$$f(-1) = 0 \text{ is a relative minimum}$$

and

$$f(\tfrac{1}{3}) = \frac{32}{27} \text{ is a relative maximum.}$$

This information enables us to give a quick sketch of the graph of f, shown in Figure 2-6e. Of course, further accuracy is obtained by plotting more points, but such quick sketches are often all we need to have.

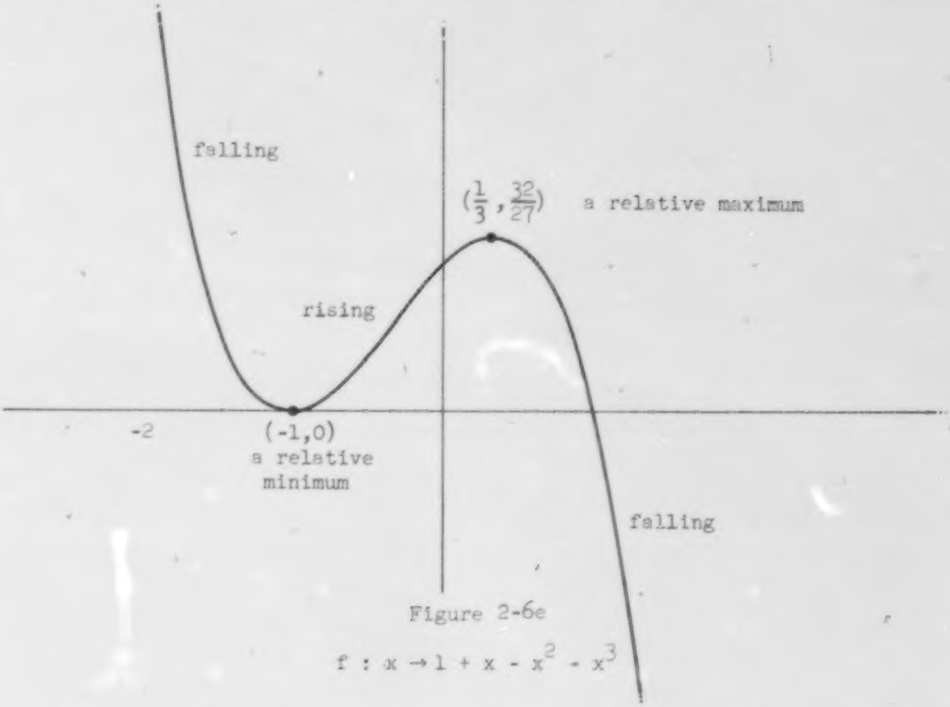


Figure 2-6e

$f : x \rightarrow 1 + x - x^2 - x^3$

Example 2-6b. The zeros of the derivative do not always lead to relative maxima and minima of a function. Consider the function

$$f : x \rightarrow x^3.$$

Its derivative is

$$f' : x \rightarrow 3x^2,$$

which has the zero $x = 0$ of multiplicity 2. The graph of f, therefore, has a horizontal tangent at $(0,0)$ but this point is not a high point or a low point on the graph of f. In fact, the graph of f crosses its tangent at $(0,0)$. Such a point is called a point of inflection. (See Figure 2-6f.) In this case

$$f'(x) > 0 \text{ if } x < 0 \text{ or if } x > 0;$$

that is, f is increasing on either side of the origin.

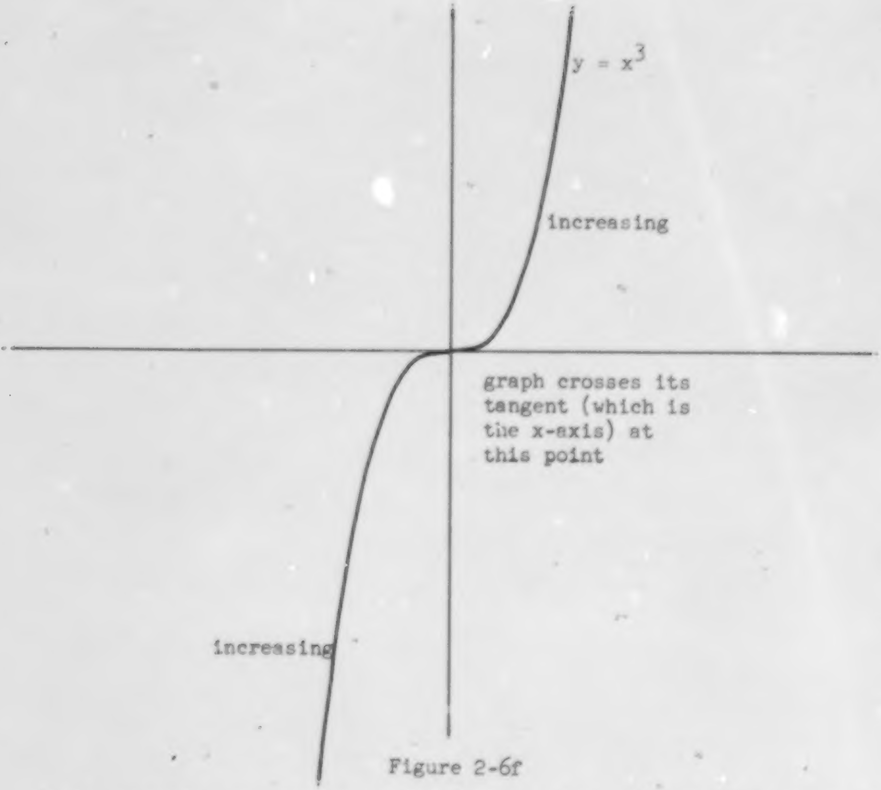


Figure 2-6f

This example serves to remind us that to determine the relative maxima and minima, we locate the critical points (that is, the zeros of f') and test the sign of f' on each side of a critical point to determine if that point is a relative maximum, minimum, or point of inflection.

Example 2-6c. Graph $f : x \rightarrow 44 + 4x - 13x^2 + 18x^3 - 9x^4$ first by plotting points from the table below Figure 2-6g and connecting them with a smooth curve, then by finding the critical points and intervals of increase and decrease.

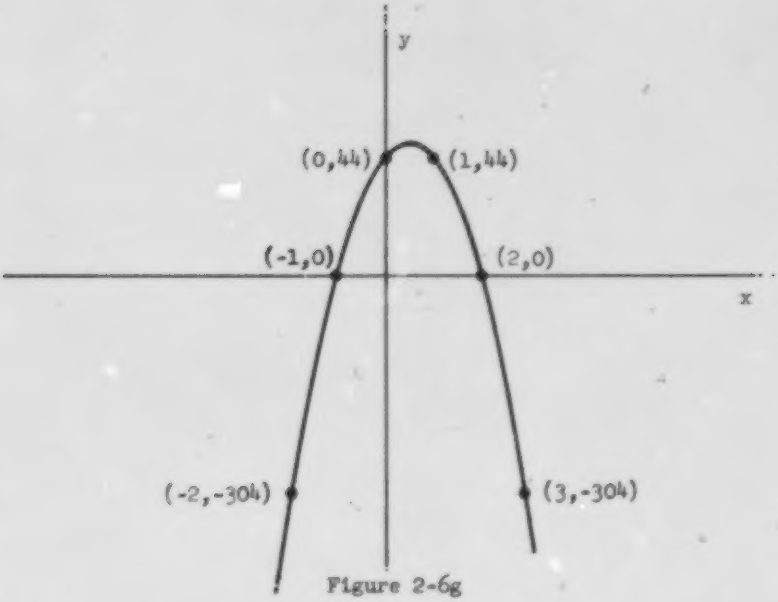


Figure 2-6g

Plotting points for

$$f : x \rightarrow 44 + 4x - 13x^2 + 18x^3 - 9x^4$$

from the table

x	-2	-1	0	1	2	3
f(x)	-304	0	44	44	0	-304

We have plotted a few points and connected them with a smooth curve in Figure 2-6g (using a compressed vertical scale for negative values of f). The graph suggests the possibility of a relative maximum point between $(0,44)$ and $(1,44)$. The derivative of f is

$$f' : x \rightarrow 4 - 26x + 54x^2 - 36x^3.$$

We suspect that $f'(x)$ will be zero somewhere in the interval $0 < x < 1$. Testing $x = \frac{1}{2}$ indeed gives

$$f'(\tfrac{1}{2}) = 0$$

so that $x - \frac{1}{2}$ is a factor of $f'(x)$. Upon factoring $x - \frac{1}{2}$ we obtain

$$f' : x \rightarrow (x - \tfrac{1}{2})(-36x^2 + 36x - 8).$$

We can factor further to obtain

$$f' : x \to -36(x - \frac{1}{2})(x - \frac{1}{3})(x - \frac{2}{3}).$$

Note that there are three critical points between $(0,44)$ and $(1,44)$. The product of three factors will be positive if and only if all three factors are positive, or exactly one is positive. Therefore,

$$f'(x) > 0 \quad \text{if} \quad x < \frac{1}{3}$$

$$f'(x) < 0 \quad \text{if} \quad \frac{1}{3} < x < \frac{1}{2}$$

$$f'(x) > 0 \quad \text{if} \quad \frac{1}{2} < x < \frac{2}{3}$$

$$f'(x) < 0 \quad \text{if} \quad x > \frac{2}{3} .$$

We conclude that

f is increasing for $x < \frac{1}{3}$ and for $\frac{1}{2} < x < \frac{2}{3}$,

while

f is decreasing for $\frac{1}{3} < x < \frac{1}{2}$ and for $x > \frac{2}{3}$.

In summary we can say that the graph of f rises until it reaches $(\frac{1}{3}, f(\frac{1}{3}))$, then falls to the point $(\frac{1}{2}, f(\frac{1}{2}))$, rises again to $(\frac{2}{3}, f(\frac{2}{3}))$ and falls beyond $(\frac{2}{3}, f(\frac{2}{3}))$. In particular we know that

$f(\frac{1}{3}) = 44\frac{4}{9}$ and $f(\frac{2}{3}) = 44\frac{4}{9}$ are relative maxima, while
$f(\frac{1}{2}) = 44\frac{7}{16}$ is a relative minimum.

We conclude that Figure 2-6g correctly indicates the increase for $x < 0$ and the decrease for $x > 1$, but is incorrect for the interval $0 < x < 1$. A more accurate representation of f on this interval is sketched in Figure 2-6h.

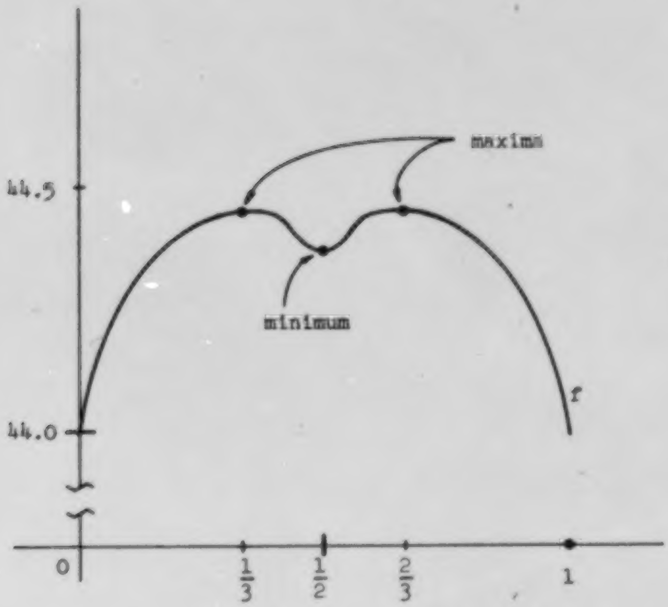


Figure 2-6h

$f : x \rightarrow 44 + 4x - 13x^2 + 18x^3 - 9x^4$ for $0 \leq x \leq 1$.

Exercises 2-6

1. Make a careful sketch on the interval $[0,1]$ (i.e., $0 \leq x \leq 1$) of the graph of the function $f : x \to 1 + x - x^2 - x^3$ given in Example 2-6a. Does the graph confirm the conclusions of the text?

2. For each of the following functions locate and characterize all extrema (maxima and minima). On what intervals is the function increasing? decreasing?

   (a) $f : x \to 4x^4 - 8x^2 + 1$

   (b) $f : x \to x^4 - 4x^3$

3. Prove that, for $x \geq 0$, $f : x \to x^2$ is an increasing function. (That is, let $x_1 > x_2 \geq 0$ and show that $x_1^2 > x_2^2$.)

4. Employing information gathered by procedures suggested in the text, sketch the graph of each of the following polynomial functions over the interval indicated using convenient scales.

   (a) $f : x \to x^3 - 3x + 1, \quad -2 \leq x \leq 2$

   (b) $f : x \to x^3 + 3x + 1, \quad -1 \leq x \leq 1$

   (c) $f : x \to x^4 - 4x^3 - 8x^2 + 64, \quad -2 \leq x \leq 5$

5. (a) Describe the behavior of the graph of $f : x \to 2x^3 - 3x^2$ on $[-1,2]$. (Maxima? Minima? Intervals of increase, decrease?)

   (b) Sketch the graph of $f : x \to 2x^3 - 3x^2$ on $[-1,2]$.

6. (a) Describe the behavior of the graph of $f : x \to -3x^4 + 8x^3$ on $[-1,3]$.

   (b) Sketch the graph of $f : x \to -3x^4 + 8x^3$ on $[-1,3]$.

7. Determine the maximum value of the function $f : x \to \dfrac{8}{x^2 - 6x + 10}$.

8. What is the greatest possible number of points where the tangent to the graph of a quadratic function $x \to Ax^2 + Bx + C$ may be horizontal?

9. Consider the function

$$f : x \to Ax^3 + Bx^2 + Cx + D, \quad A \neq 0.$$

(a) Find f'.

(b) What is the maximum number of zeros that f' can have?

(c) How many relative extrema (maxima and minima) can f have?

(d) If the graph of f has a relative maximum point, must it have a relative minimum point? Explain or give examples.

(e) If $f'(x_1) = f'(x_2) = 0$, determine $\frac{x_1 + x_2}{2}$.

10. Consider the functions

$$f : x \to (x + 1)^3(x + 2)$$

and

$$g : x \to (x + 1)^2(4x + 7).$$

(a) How are the functions related?

(b) Sketch the graphs of f and g on the same set of axes.

11. Suppose that x_1 and x_2 are zeros of

$$f : x \to Ax^2 + Bx + C, \quad A > 0.$$

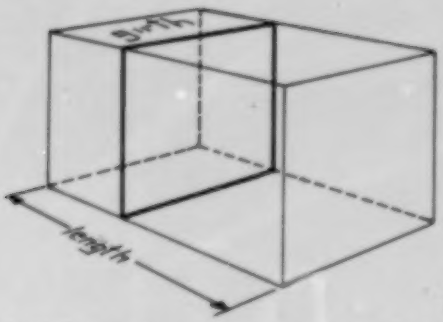
Show that f has a minimum at

$$x = \frac{x_1 + x_2}{2}.$$

12. Determine the relative maximum and minimum points of the graph of $f : x \to 3x^4 - 12x^3 + 12x^2 - 4$.

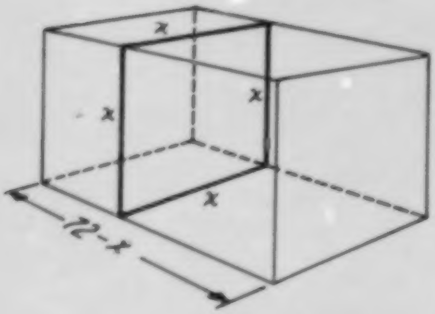
2-7. Optimization Problems

The post office limits the size of a parcel post package by requiring that its length plus its girth may not exceed 72 inches.



Is there a package, subject to such a restriction, which has greater volume than all other such packages? If so, what are its dimensions?

In this form, the problem is hard to handle. Suppose we simplify it by asking if there is a (rectangular) package with a square cross-section which has maximum volume subject to postal regulations. If we let x represent the width in inches of the square cross-section, then the girth of the package is 4x inches and its length is at most 72 - 4x inches, according to the post office.



Hence, the volume in cubic inches of the package is at most $V = x \cdot x \cdot (72 - 4x)$. This formula defines a polynomial function

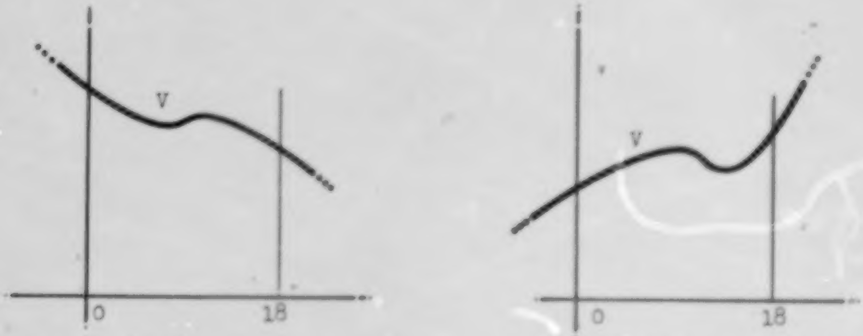
$$(1) \qquad V : x \to 72x^2 - 4x^3,$$

and we wish to find its maximum value. Note that $V(x)$ has no maximum if we do not restrict x: if $x < 0$, $V(x)$ is positive and gets arbitrarily large as x becomes negatively infinite. However, this causes no difficulty because we are interested only in values of x between 0 and 18, because only in this interval $[0,18]$ are all the dimensions of the parcel post package sensible positive lengths.

Hence, our idealized model of the problem is:

> Find the maximum value of the function $V : x \to 72x^2 - 4x^3$ for x in $[0,18]$.

The desired maximum may occur at one of the end points of the closed interval $[0,18]$, if for example, the graph of V looked like one of the curves shown in Figure 2-7a.



Max. in $[0,18]$ at $x = 0$. Max. in $[0,18]$ at $x = 18$.

Figure 2-7a

However, if the maximum value $V(x)$ occurs between the endpoints of the interval $[0,18]$, i.e., for some $x = a$ in the open interval $(0,18)$, then the remarks in Section 2-6 show that $V'(a) = 0$. If $V'(a) > 0$, there would be higher points immediately to the right of a, and if $V'(a) < 0$, there would be higher points immediately to the left of a.

Thus the maximum volume for our parcel post Package occurs where $V'(x) = 0$, or where $x = 0$ or $x = 18$.

$$\text{If } V'(x) = 12x(12 - x) = 0 \text{ then } x = 0 \text{ or } x = 12.$$

The possible maximum volumes are therefore

$$V(0) = 72(0)^2 - 4(0)^3 = 0$$

$$V(12) = 72(12)^2 - 4(12)^3 = 3{,}456$$

$$V(18) = 72(18)^2 - 4(18)^3 = 0.$$

Clearly, $V(12)$ is the largest of these and by the above remarks it must be the relative maximum of V on the interval $[0,18]$. Hence, the dimensions of the most voluminous parcel of this sort acceptable to the post office are

width = x inches = 12 inches,
height = x inches = 12 inches,
length = $72 - 4x$ inches = 24 inches,

and its volume is 3,456 cubic inches.

The following examples do not begin to indicate the wide range of possible applications of the method used above.

Example 2-7a. A man proposes to make an open box by cutting a square from each corner of a piece of cardboard 12 inches square and then turning up the sides. Find the dimensions of each square he must cut in order to obtain a box with maximum volume.

Let the side of the square be cut out be x inches. The base of the box will be $12 - 2x$ inches on each side and the depth will be x inches.

The volume V in cubic inches will be

$$V = (12 - 2x)(12 - 2x)(x)$$

$$= 144x - 48x^2 + 4x^3.$$

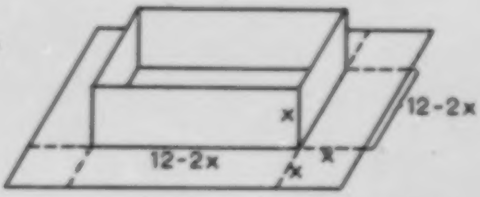


Figure 2-7b

We suppose that $0 \leq x \leq 6$, for otherwise V will be negative. Our problem is to maximize

$$f : x \to 144x - 48x^2 + 4x^3$$

subject to the condition $0 \leq x \leq 6$. The derivative is

$$f' : x \to 144 - 96x + 12x^2 = 12(6 - x)(2 - x)$$

so the zeros of f' are 2, 6. We know that this maximum must occur at one of the points

$$c = 0, c = 2, \text{ or } c = 6.$$

We find that $f(0) = f(6) = 0$ and $f(2) > 0$, so that $f(2)$ must be the largest value of f on the interval $0 \leq x \leq 6$. With a 2 inch square cut from each corner, the box will have dimensions $8 \times 8 \times 2$. Since $f(2) = 8 \times 8 \times 2 = 128$, the maximum volume is 128 cubic inches.

Example 2-7b. We wish to plant one square and one circular flower bed, surrounding them with 15 yards of fencing. What should be the dimensions of the two fences so as to contain flower beds of greatest possible area?

Let s be the side of the square bed and r the radius of the circular bed. Denote the sum of the areas of the two beds by A. Then

$$A = s^2 + \pi r^2 \tag{2}$$

and

$$4s + 2\pi r = 15. \tag{3}$$

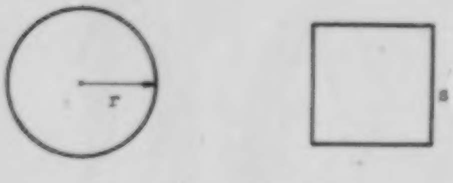


Figure 2-7c

Solving (3) for s and substituting this into (2) gives our area in terms of the circular radius r:

$$\begin{aligned} A &= \left(\frac{15 - 2\pi r}{4}\right)^2 + \pi r^2 \\ &= \left(\frac{\pi^2 + 4\pi}{4}\right) r^2 - \frac{15\pi}{4} r + \frac{225}{16}. \end{aligned}$$

We can suppose that

$$0 \leq r \leq \frac{15}{2\pi} ,$$

the endpoints of this interval corresponding to the respective situations of no circular bed and no square bed. Thus we seek to maximize the function

$$f : r \rightarrow (\frac{\pi^2 + 4\pi}{4})r^2 - \frac{15\pi}{4} r + \frac{225}{16}$$

over the interval $0 \leq r \leq \frac{15}{2\pi}$. The derivative is

$$f' : r \rightarrow (\frac{\pi^2 + 4\pi}{2})r - \frac{15\pi}{4} .$$

Solving $f'(r) = 0$ gives

$$r = \frac{15}{2\pi + 8} .$$

Hence, f must have its maximum value on the interval $0 \leq r \leq \frac{15}{2\pi}$ at one of the points

$$c = 0, \; c = \frac{15}{2\pi + 8} \text{ or } c = \frac{15}{2\pi} .$$

We examine the values of f for each of these values of c:

$$f(0) = \frac{225}{16} \approx 14.06$$

$$f(\frac{15}{2\pi + 8}) = 7.88$$

and

$$f(\frac{15}{2\pi}) \approx 17.90.$$

We see that the maximum value of

$$f : r \rightarrow (\frac{\pi^2 + 4\pi}{4})r^2 - \frac{15\pi}{4} r + \frac{225}{16}$$

subject to the restriction $0 \leq r \leq \frac{15}{2\pi}$ is attained at the right endpoint $\frac{15}{2\pi}$.

Our conclusion is that the problem has no solution in the terms posed; a square and a round flower bed together will never encompass as great an area as a single round bed whose perimeter equals the total length available.

Similar problems requiring the minimization of a function over an interval often occur, and can be solved similarly. If f is a polynomial function defined on some interval $[a,b]$, the minimum value $f(x)$ occurs either at $x = a$, $x = b$, or where $f'(x) = 0$. To justify this claim we need only observe that if the minimum occurs at $x = c$, $a < c < b$, then if $f'(c) > 0$ there are lower points immediately to the left of c, and if $f'(c) < 0$ there are lower points immediately to the right of c.

Example 2-7c. What should be the dimensions of the flower beds in Example 2-7b so that the least possible area is encompassed?

Following the analysis of Example 2-7b, we wish to minimize the function

$$f : r \rightarrow \left(\frac{\pi^2 + 4\pi}{4}\right)r^2 - \frac{15\pi}{4} r + \frac{225}{16}$$

over the interval $[0, \frac{15}{2\pi}]$. $f(r)$ is minimum either at $r = 0$, $r = \frac{15}{2\pi}$ or where $f'(r) = 0$. In Example 2-7b we found that $f'(r) = 0$ if and only if

$$r = \frac{15}{2\pi + 8} .$$

Hence, the possible minimum values for f are

$$f(0) \approx 14.06$$

$$f\left(\frac{15}{2\pi + 8}\right) \approx 7.88$$

$$f\left(\frac{15}{2\pi}\right) \approx 17.90$$

We see that the minimum value of f occurs when $r = \frac{15}{2\pi + 8}$. Hence, the combination square and circular garden surrounded by 15 yards of fencing has the least area when the circle has radius $r = \frac{15}{2\pi + 8} \approx 1.05$ yards and the square has side $s = \frac{1}{4}(15 - 2\pi r) \approx 2.10$ yards.

Example 2-7d. Find the point on the graph of $y = x^2$ that is nearest the point $A(3,0)$.

Recall that the distance between (x_1,y_1) and (x_2,y_2) is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The distance from $A(3,0)$ to a point $P(x,x^2)$ on the graph of $y = x^2$ is thus given by

$$AP = \sqrt{(x - 3)^2 + (x^2)^2}.$$

Our problem is to choose P so that this distance AP is least.

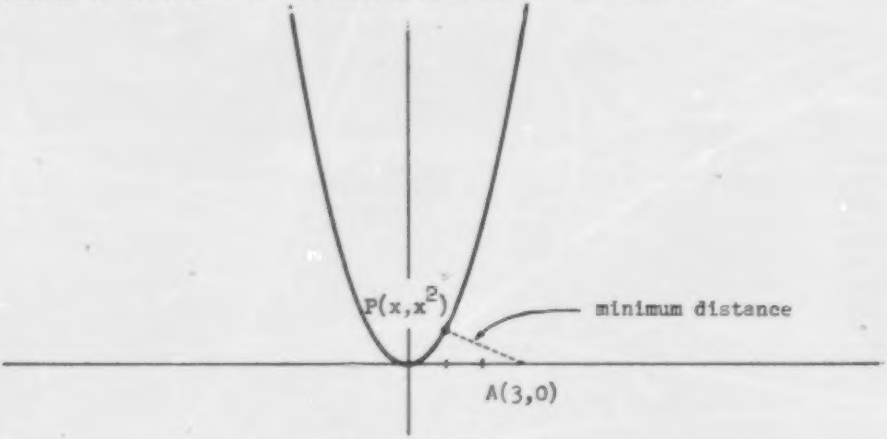


Figure 2-7d

This expression for the distance AP is not a polynomial so our techniques cannot be directly applied. Note, however, that

$$\begin{aligned}(AP)^2 &= (x - 3)^2 + (x^2)^2 \\ &= 9 - 6x + x^2 + x^4\end{aligned}$$

which is a polynomial expression. Furthermore, if x is such that $(AP)^2$ is least, then AP will also be least. Thus we need only choose x so that

$$f : x \to 9 - 6x + x^2 + x^4$$

has its least value.

The derivative of f is

$$f' : x \to -6 + 2x + 4x^3 = (x - 1)(4x^2 + 4x + 6).$$

The factor $4x^2 + 4x + 6$ is always positive, since we can complete the square to obtain

$$\begin{aligned}4x^2 + 4x + 6 &= 4x^2 + 4x + 1 + 6 - 1 \\ &= (2x + 1)^2 + 5 \geq 5.\end{aligned}$$

Thus, $f'(x) < 0$ if $x < 1$ and $f'(x) > 0$ if $x > 1$. Therefore, by the remarks in Section 2-6 we know that f has a relative minimum at $x = 1$. Hence, the point $(1,1)$ is the point of the graph $y = x^2$ which is closest to $A(3,0)$.

Exercises 2-7

1. Consider the function

$$f : x \to 3x^4 + 4x^3 - 12x^2 + 5.$$

Determine the behavior and sketch the graph of f.

2. Find the extrema (maxima and minima) of the function

$$f : x \to 4x^5 - 5x^4 - 40x^3 + 100$$

on the interval $-3 \leq x \leq 4$. Sketch the graph of f.

3. A ball is thrown upward so that its height t seconds later is s feet above the earth, where

$$s = 96t - 16t^2.$$

What is the maximum height the ball will reach?

4. Show that of all the rectangles having a given perimeter p, the square has the largest area.

5. Sketch the graph of the function

$$f : x \to -4x^3 + 72x^2$$

over the interval $[0,18]$, indicating extrema.

6. A rectangular box with square base and open top is to be made from a 20 ft. square piece of cardboard. What is the maximum volume of such a box?

7. A rectangular field is to be adjacent to a river and is to have fencing on three sides, the side on the river requiring no fencing. If 100 yards of fencing is available, find the dimensions of the field with largest area.

8. The sum of two positive numbers is N. Determine the numbers so that the product of one and the square of the other will be a maximum.

9. A wire 24 inches long is cut in two, and then one part is bent into the shape of a circle and the other into the shape of a square. How should it be cut if the sum of the areas is to be a minimum?

10. Given the requirements of Number 9, determine how the wire should be cut if the sum of the areas is to be a maximum.

11. A four-ft.-wire is to be cut into two pieces: one piece to become the perimeter of a square, the other the circumference of a circle. Determine how it should be cut to enclose

    (a) minimum area;

    (b) maximum area.

12. Determine the dimensions of the rectangle with perimeter 72 feet which will enclose the maximum area.

13. Determine the radius and height of the right circular cylinder of greatest volume that can be inscribed in a right circular cone with radius r and height h.

14. A man has 600 yds. of fencing which he is going to use to enclose a rectangular field and then subdivide the field into two plots with a fence parallel to one side. What are the dimensions of such a field if the enclosed area is to be a maximum?

15. An open box is to be made by cutting out squares from the corners of a rectangular piece of cardboard and then turning up the sides. If the piece of cardboard is 12" by 24", what are the dimensions of the box of largest volume made in this way?

16. A rectangle has two of its vertices on the x-axis and the other two above the axis on the graph of the parabola $y = 16 - x^2$. What are the dimensions of such a rectangle if its area is to be a maximum?

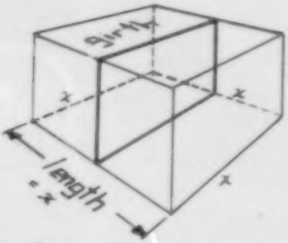
17. A stone wall 100 yards long stands on a ranch. Part or all of it is to be used in forming a rectangular corral, using an additional 260 yards of fencing for the other three sides. Find the maximum area which can be so enclosed.

18. Find the point on the graph of the equation $y^2 = 4x$ which is nearest to the point $(2,1)$.

19. Find the dimensions of the right circular cylinder of maximum volume inscribed in a sphere of radius 10 inches.

20. What number most exceeds its square?

21. Suppose that the base of the parcel post package mentioned in the text is taken to be square.



Find the package of this shape which has maximum volume subject to the postal restriction that the sum of its length and girth may not exceed 72 inches.

22. A rectangle has two of its vertices on the x-axis and the other two above the axis on the parabola $y = 6 - x^2$. What are the dimensions of such a rectangle if its area is to be a maximum?

23. A rectangular sheet of galvanized metal is bent to form the sides and bottom of a trough so that the cross section has this shape: $\sqcup$

If the metal is 14 inches wide how deep must the trough be to carry the most water?

24. A rectangular sheet of galvanized metal is to be made into a trough by bending it so that the cross section has a $\sqcup$ shape. If the metal is 10 inches wide, how deep must the trough be to carry the most water?

25. Prove that with a fixed perimeter P the rectangle which has a maximum area is a square.

26. Determine the area of the largest rectangle that can be inscribed in the region bounded by the graphs of $y^2 = 8x$ and $x = 4$.

27. Show that there is no point on the ellipse given by $x^2 + 4y^2 = 8$ closer to the point $(1,0)$ than $(\frac{4}{3}, \frac{1}{3}\sqrt{14})$.

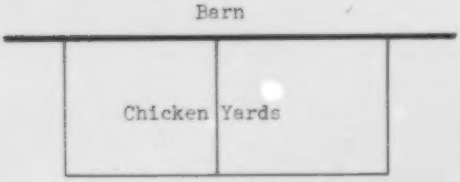
28. Find the altitude of the cone of maximum volume that can be inscribed in a sphere of radius r.

29. A rectangular pasture, with one side bounded by a straight river, is fenced on the remaining three sides. If the length of the fence is 400 yards, find the dimensions of the pasture with maximum area.

30. A farmer plans to enclose two chicken yards next to his barn with fencing, as shown. Find

    (a) the maximum area he can enclose with 120 feet of fence;

    (b) the maximum area he can enclose if the dividing fence is parallel to the barn.



In the following problems (Nos. 31-35) meaningful replacements for the variables are obviously restricted to positive integers, but we must consider the functions to be continuous in order to apply the techniques of this chapter.

31. A printer will print 10,000 labels at a base price of $1.50 per thousand. For a larger order the base price on the entire lot is decreased by 3 cents for each thousand in excess of 10,000. For how many labels will the printer's gross income be a maximum?

32. A manufacturer can now ship a cargo of 100 tons at a profit of $5.00 per ton. He estimates that by waiting he can add 20 tons per week to the shipment, but that the profit on all that he ships will be reduced 25¢ per ton per week. How long will it be to his advantage to wait?

33. A peach orchard now has 30 trees per acre, and the average yield is 400 peaches per tree. For each additional tree planted per acre, the average yield is reduced by approximately 10 peaches. How many trees per acre will give the largest crop of peaches?

34. A potato grower wishes to ship as early as possible in the season in order to sell at the best price. If he ships July 1st, he can ship 6 tons at a profit of $2.00 per ton. By waiting he estimates he can add 3 tons per week to his shipment but that the profit will be reduced by $\frac{1}{3}$ dollar per ton per week. When should he ship for a maximum profit?

35. A real estate office handles 80 apartment units. When the rent of each unit is \$60.00 per month, all units are occupied. If the rent is increased \$2.00 a month, on the average one further unit remains unoccupied. Each occupied unit requires \$6.00 worth of service a month (i.e., repairs and maintenance). What rent should be charged in order to obtain the most profit?

36. A right triangle with hypotenuse k is rotated about one of its legs. Find the maximum volume of the right circular cone produced.

37. Determine the dimensions of the rectangle with greatest area which can be inscribed in a circle of radius R.

38. Determine the dimensions of the rectangle with greatest perimeter which can be inscribed in a circle of radius R.

2-8. Rate of Change: Velocity and Acceleration

The derivative f' of a polynomial function f has been defined as the function whose value at a is the slope of the tangent line to the graph of f at the point $(a,f(a))$.

In many physical situations the value $f'(a)$ can also be interpreted as velocity. Let us look at an example.

Suppose a solid ball is dropped from a 2000 foot tower. Let s denote its distance (in feet) from the top of the tower at time t (in seconds) after it is released. Experimentation has shown that s is approximately related to t by the equation

$$s = 16t^2.$$

Thus, we sometimes say that the fallen distance (s feet) is a function of time (t seconds). More precisely, the equation $s = 16t^2$ specifies the function $f : t \to 16t^2$.

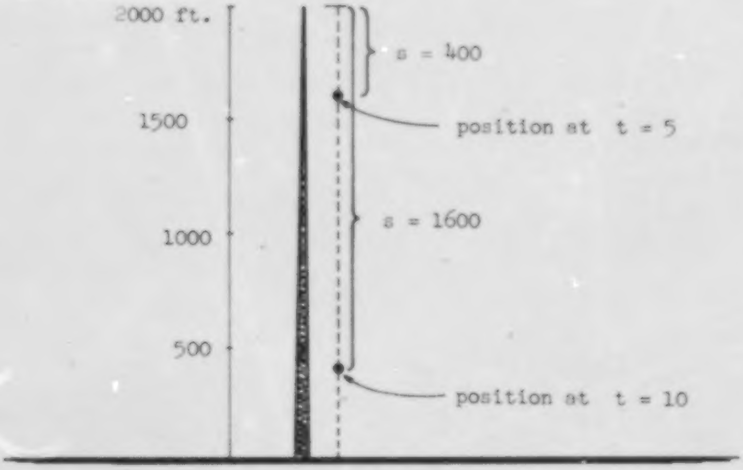


Figure 2-8a

We wish to formulate a suitable concept of velocity, so that we can answer questions such as: How fast is the ball falling after it has fallen 5 seconds? To do this we first define the average velocity in the time interval $t_1 \leq t \leq t_2$ as the ratio

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1} \text{ feet/second.}$$

This is just the ratio of the distance traveled in the time interval to the length of the interval.

For example, in the time interval $[4.5, 5]$ (i.e., $4.5 \leq t \leq 5$) the ratio is

$$\frac{16 \cdot (5)^2 - 16 \cdot (4.5)^2}{5 - 4.5} = \frac{400 - 324}{0.5} = 152.$$

Therefore, the average velocity between 4.5 and 5 seconds is 152 ft./sec. In the time interval $[5, 5.2]$ the ratio is

$$\frac{16 \cdot (5.2)^2 - 16 \cdot (5)^2}{5.2 - 5} = \frac{432.64 - 400}{0.2} = 163.2.$$

Whence the average velocity between 5 and 5.2 seconds is 163.2 ft./sec.

Suppose that h is a small positive quantity. The average velocity in the time interval $5 \leq t \leq 5 + h$ is then

$$\frac{f(5 + h) - f(5)}{h} \text{ ft./sec.}$$

This is just our old friend, the difference quotient used in approximating the derivative. We know that as the time interval becomes shorter, h approaches zero and the ratio expressing average velocity approaches $f'(5)$, the value of the derivative of the distance function f at $t = 5$. We therefore, adopt the following definition:

> The (instantaneous) velocity of a body whose position after t seconds is given by $f(t)$, is $f'(t)$.

In our example, $f' : t \to 32t$ and $f'(5) = 160$, so that the velocity after 5 seconds is 160 ft./sec.

In summary, the function

$$f : t \to s = 16t^2$$

describes the position of the ball at time t, while

$$f' : t \to v = 32t$$

describes its velocity at time t.

Velocity is a measure of how the position of a moving body changes over time. It is often characterized as a rate of change of position with respect to time. The acceleration of a moving body is also a rate of change -- it measures how the velocity is changing over time.

Acceleration

We have seen that velocity, like distance, can be expressed as a function of time. As a solid ball falls from a tower, for example, its velocity (in ft./sec.) after t seconds is given by

$$v = 32t.$$

This formula specifies a velocity function

$$g : t \to 32t,$$

and allows us to determine how the velocity of the falling body is changing over time. The rate of change of velocity with respect to time is called acceleration.

Just as we defined average velocity over a time interval, we can define average acceleration over a time interval $[t_1,t_2]$ as the ratio of the change in velocity to the length of time $t_2 - t_1$:

$$\frac{g(t_2) - g(t_1)}{t_2 - t_1}.$$

In the time interval $[4.5,5]$ the ratio is

$$\frac{32(5) - 32(4.5)}{5 - 4.5} = \frac{32(0.5)}{0.5} = 32.$$

Hence, the average acceleration of a falling body between 4.5 and 5 seconds of lapsed time is 32 ft./sec. per second.

In a short time interval $[t,t + h]$, the expression for the average acceleration is

$$\frac{g(t + h) - g(t)}{h}.$$

As the time interval becomes shorter, h approaches 0 and the ratio expressing average acceleration approaches $g'(t)$, the derivative of the velocity function. As before, we therefore define:

> The (instantaneous) acceleration of a body whose velocity after t seconds is given by $g(t)$, is $g'(t)$.

In our example, $g : t \to 32t$ and $g' : t \to 32$, indicating that the acceleration of a falling body in the absence of air resistance is a constant approximately equal to 32 ft./sec. each second.

In Chapter 7 we shall see that constant acceleration of an object (e.g., $a = 32$) guarantees that its velocity is a linear function of time, (e.g., $v = 32t$) and that the distance it travels is a quadratic function of time, (e.g., $s = 16t^2$).

Example 2-8a. What is the velocity of the ball dropped from the top of the 2000 foot building at the time it strikes the ground?

The distance function is

$$f : t \to 16t^2$$

and the ball is dropped from a height of 2000 feet. Setting

$$16t^2 = 2000,$$

we see that the ball strikes the ground when

$$t = \sqrt{\frac{2000}{16}} = 5\sqrt{5}.$$

Since the velocity function is

$$f' : x \to 32t,$$

we find $f'(5\sqrt{5}) \approx 357.8$. Therefore the (impact) velocity after $5\sqrt{5}$ sec. is approximately

357.8 ft/sec.

Example 2-8b. A car is being driven at the rate of 60 mi./hr. (88 ft./sec.) when the brakes are uniformly applied until the car comes to a complete stop. Suppose that the function

$$f : t \to 88\left(t - \frac{t^2}{10}\right)$$

describes the distance $s = f(t)$ in feet traveled in t seconds after the brakes are applied.

(i) How many feet does the car move before it stops?

(ii) Show that the acceleration is negative and constant.

The velocity function is

$$f' : t \to 88 - \frac{88}{5}t$$

and the car will stop at the point where the velocity is 0. Solving $f'(t) = 0$ gives $t = 5$. Since $f(5) = 220$ the distance traveled in 5 seconds is 220 feet.

The derivative of the velocity function $g : t \to 88 - \frac{88}{5}t$ (renamed for convenience) is the acceleration function

$$g' : t \to -\frac{88}{5} .$$

Therefore, the acceleration after t seconds is $-\frac{88}{5}$ feet per second per second, which is indeed constant negative acceleration (deceleration). This reflects the physical fact that the car is slowing down due to the application of the brakes. The fact that the acceleration is constant is based on the assumption that braking is uniform. (Due to brake fade, the pressure applied to the brakes must increase to maintain a uniform deceleration.)

It is often useful in other situations to interpret the value of the derivative as a rate of change. Given a polynomial function $f : x \to f(x)$, the average rate of change of f in the interval $[a,b]$ is defined to be

$$\frac{f(b) - f(a)}{b - a} .$$

Since the limit of this as the length of the interval approaches zero (b approaches a) is $f'(a)$, it is appropriate to refer to the derivative f' of f as the (instantaneous) rate of change of f at the point $x = a$. This is consistent with our interpretation of $f'(a)$ as the slope of the tangent line at $x = a$, for the slope of a line does measure its rate of rise (or fall). The tangent line is the line of "best fit" near $(a,f(a))$ and hence its slope (rate of change) gives a measure of the rate of change of f at that point.

Example 2-8c. The volume of a sphere is a function of its radius. Find the rate of change of the volume with respect to the radius. What is this rate of change when the radius is 6 inches?

Letting V denote the volume (in cubic inches) and r the radius (in inches), we have

$$V = \frac{4}{3}\pi r^3.$$

In other words, if we let f denote the function $f : r \to \frac{4}{3}\pi r^3$, the volume V is given by $V = f(r)$ (in cubic inches). The derivative of f is

$$f' : r \to (\frac{4}{3}\pi)(3r^2) = 4\pi r^2.$$

The rate of change of the volume V with respect to the radius r is thus $4\pi r^2$; when $r = 6$, the volume is changing at the rate of

$$144\pi \text{ in.}^2/ \text{ unit change in radius.}$$

Speed

We have defined velocity as the (instantaneous) rate of change of the position of an object with respect to time. This definition implies that velocity has direction, because it involves not only how fast the object is moving, but also from where to where. For motion along a straight line then, the velocity of a moving object is a signed quantity: it is positive if the motion is in the direction we define to be positive, and it is negative if the motion is in the opposite direction. What we intuitively think of as the speed of the object is independent of direction. We define speed to be the absolute value of velocity. The following familiar example (Exercises 1-3, No. 12) will make the distinction clear.

Example 2-8d. Suppose that a rock is thrown straight up from the ground with an initial speed of 64 ft./sec. Contrary to our analysis of the ball dropped from the 2000 foot tower, let us take "up" as the positive direction and "down" as the negative direction. Hence, the initial velocity of our rock is +64 ft./sec.

We know that if the rock were not acted upon by gravity the position of the rock after t seconds would be given by the function

$$f : t \to 64t.$$

But the force of gravity acts on the rock, just as on the ball dropped from the tower, adding a downward (in this case, negative) component, $-16t^2$, to the position of the rock at time t. Hence, the function

$$f : t \to 64t - 16t^2$$

describes the position of the rock at time t, and

$$f': t \to 64 - 32t$$

gives the velocity of the rock after t seconds. We note that

for $t < 2$, $f'(t) > 0$,

for $t = 2$, $f'(t) = 0$,

and for $t > 2$, $f'(t) < 0$,

indicating that between 0 and 2 seconds the rock is rising (moving in the chosen positive direction) while after 2 seconds have elapsed the rock is falling (moving in the negative direction).

At $t = 1.5$ seconds the velocity of the rock is

$$f'(1.5) \text{ ft./sec.} = 12 \text{ ft./sec.}$$

At $t = 2.5$ seconds, the velocity is

$$f'(2.5) \text{ ft./sec.} = -12 \text{ ft./sec.}$$

We conclude that the speed of the rock is the same at $t = 1.5$ and $t = 2.5$, namely 12 ft./sec. Similarly, the initial speed and the final (impact) speed of the rock are the same since the initial velocity is +64 ft./sec. (up) and the impact velocity is -64 ft./sec. (down).

Renaming the velocity function

$$g : t \to 64 - 32t,$$

we see that the acceleration of the rock is given by

$$g': t \to -32,$$

indicating that gravity accelerates the rock at the rate of 32 ft./sec. each second -- downward, of course. Note that this negative acceleration decreases the rock's velocity constantly; on the way up the decrease in velocity amounts to a slowing of the speed of the rock, while on the way down the "decrease" in velocity (becoming more negative) creates an increase in the speed of the rock.

Exercises 2-8

1. (a) Determine the rate of change of the area of a circle with respect to its radius r. Compare your result with the formula for the circumference C of a circle in terms of radius r.

   (b) What is the rate of change of the volume of a sphere with respect to its radius r? Compare this result with the formula for the surface area s of a sphere in terms of radius r.

2. A certain motion is described, from the time $t = 5$ until the time $t = 8$, by the equation

$$s = f(t) = 2t^3 - 39t^2 + 252t - 535.$$

   (a) We submit that if the distance s at time t is given by

$$s = f(t) = 2t^3 - 39t^2 + 252t - 535,$$

   then the velocity v at time t is given by

$$v = 6t^2 - 78t + 252.$$

   Explain why this is true.

   (b) Sketch the graph of the function

$$f : t \rightarrow s = 2t^3 - 39t^2 + 252t - 535$$

   on the interval $5 \leq t \leq 8$.

   (c) Sketch the graph of

$$f' : t \rightarrow v = 6t^2 - 78t + 252$$

   on the same interval $[5,8]$.

   (d) Determine the zeros of f'.

   (e) When does the particle whose motion is being described come to rest for an instant as it shifts direction?

   (f) When is the particle the greatest distance from its starting point on $[5,7]$?

   (g) What is the greatest distance of the particle from its starting point on $[5,7]$?

   (h) When is the next time on $[5,8]$ when the distance of the particle from its starting point is as great as its greatest distance on $[5,7]$?

(i) When is the particle the greatest distance from its starting point on $[5,8]$?

(j) When is the speed of the particle the greatest on $[6,7]$?

3. What is the acceleration at time t of the particle whose motion is considered in Number 2?

4. Recall Example 2-8b.

(a) For a car traveling at 60 mi./hr., how many seconds are required after the brakes are applied (and held) before the car comes to a complete stop?

(b) How far will a car traveling at 60 mi./hr., go after the brakes are applied?

(c) Suppose at time $t = 0$ the brakes are applied on a car moving at velocity 60 mph and kept on until the car is brought to a stop producing a constant deceleration (negative acceleration) of α ft./sec^2. Given an approximation for α to ensure that the car will continue to travel only 100 ft. after the brakes are applied.

(d) Using your approximation for α from part (c), determine the distance the car would require to stop if it were traveling at 30 mph.

5. Let us assume that a pellet is projected straight up and after awhile comes straight down via the same vertical path to the place on the ground from which it was launched. After t seconds the pellet is s feet above the ground. Some of the ordered pairs (t,s) are given in the following table.

t	0	1	2	3	4	5	6	7	8	9	10
s	0	144	256	336	384	400	384	336			0

We shall intentionally avoid certain physical considerations such as air resistance. Moreover, we shall deal with simple numbers rather than quantities measured to some prescribed degree of accuracy which might arise from the data of an actual projectile problem in engineering.

(a) Interpolate from the data given to determine the height of the projectile after eight and nine seconds respectively. (Guess, using symmetry as your guide.) Does extrapolation to find values of s for $t = -1$ or $t = 11$ make sense on physical grounds? After how many seconds does the projectile appear to have reached its maximum height? What seems to be the maximum height?

(b) Does s appear to be a function of t? If so, discuss the domain and range, taking physical considerations into account.

(c) If we were to plot a graph of $s = f(t)$,

  (1) is it plausible on physical grounds to restrict our graph to the first quadrant?

  (2) Does the data suggest that the scale on the s-axis (vertical) should be the same as the scale on the t-axis (horizontal)?

(d) Keeping in mind your responses to part (c), plot the ordered pairs (t,s) from the table. Connect the points with a smooth curve. What is the name of the function suggested by the graph? On physical grounds is it feasible that there would be a real value of s for every real number assigned to t over the interval $0 \leq t \leq 10$? Were we probably justified in connecting the points?

(e) Assuming that the equation $s = f(t) = At^2 + Bt + C$ was used to develop the entries in our table, find values for constants A, B, and C.

(f) Sketch the graph given by the equation $s = 160t - 16t^2$ over the interval $0 \leq t \leq 10$. Using a more carefully plotted graph of the above set, connect the point where $t = 1$ with the point where $t = 2$ with a chord. What is the slope of this chord? Estimate the slope of the curve at $t = 1$ and $t = 2$.

(g) If the units of s are feet and the units of t are seconds, what are the units of slope? What word is commonly associated with this ratio of units? What would you guess are the physical interpretations of positive, zero, and negative values of this ratio?

(h) Draw the graph of $v = 160 - 32t$ over the interval $0 \leq t \leq 10$. Compare the values of v for $t = 1$ and $t = 2$ respectively with your estimates for the slopes of the graph of $s = 160t - 16t^2$ in part (f).

(i) Average the values of v for $t = 1$ and $t = 2$ and compare this average with the slope of the chord connecting the points where $t = 1$ and $t = 2$ in part (f).

(j) If the units of v are ft./sec. and the units of t are seconds, what are the units of the slope of the line $v = 160 - 32t$? What word from physics is commonly associated with this ratio of units? Does the minus sign along with the particular numerical value of this slope have any special connotation from your experience?

6. A projectile is fired straight up and after awhile comes straight down via the same vertical path to the place on the ground from which it was launched. After t seconds the projectile is $s = 160t - 16t^2$ feet above the ground.

(a) After how many seconds does the projectile strike the ground?

(b) What is the velocity of the projectile after t seconds?

(c) What is the initial velocity?

(d) What is the impact velocity?

(e) How high is the projectile after 4 seconds?

(f) How high is the projectile after 6 seconds?

(g) After how many seconds does the projectile reach its maximum height?

(h) How high does the projectile go?

(i) How far has the projectile traveled after 6 seconds?

7. A ball is thrown upward from the ground so that after t seconds its height s feet is given by the function

$$f : t \rightarrow s = 96t - 16t^2 .$$

(a) The path of the ball is straight up and straight down. What is the graph of the function f?

(b) What is the derivative of f? What is the velocity function?

(c) How high is the ball after one second?

(d) How high is the ball after 5 seconds?

(e) How far has the ball traveled after 5 seconds?

(f) What is the initial velocity of the ball?

(g) How long is the ball in the air?

(h) What is the impact velocity when the ball strikes the ground?

(i) What is the constant acceleration acting upon the ball?

(j) Give a distance function $g : t \to s$, where s is the number of feet above the ground after t seconds, appropriate for the situation if the ball were thrown straight upward with an initial velocity of 96 ft./sec. from a tower 200 ft. high.

(k) Give a distance function $G : t \to s$, where s is the number of feet above the ground after t seconds appropriate for the situation if the ball were thrown straight downward with an initial velocity of 96 ft./sec. from a tower 200 feet high.

(*l*) Give a distance function $F : t \to s$, where s is the number of feet displacement from the top of the tower after t seconds if the ball is simply dropped from the top of a tower.

8. The velocity of an object, whose location on a straight line at time $t = t_0$ is given by $s = f(t)$, is the limit of the ratio

$$\frac{f(t) - f(t_0)}{t - t_0}$$

as t approaches t_0. This limit is the value of the derivative f' at $t = t_0$. Experimentally it has been established that the distance covered in time t by a freely falling body is proportional to t^2, and therefore it can be represented by the function $f' : t \to ct^2$, where c is a positive constant. Show that the velocity of a freely falling body is directly proportional to the time.

9. Suppose a projectile is ejected with initial velocity of v_0 feet per second, at a point P which is 20 feet above the ground. Neglect friction and assume that the projectile moves up and down in a straight line. Let $f(t)$ denote the height (above P) in feet that the projectile attains t seconds after ejection. Note that if gravitational attraction were not acting on the projectile, it would continue to move upward with a constant velocity, traveling a distance of v_0 feet each second, so that its height at time t would be given by $f(t) = v_0t$. We know that the force of gravity acting on the projectile causes it to slow down until its velocity is zero and then travel back to the earth. On the basis of

physical experiments the formula $f(t) = v_0 t - \frac{g}{2} t^2$, where g represents the acceleration of gravity, is used to represent the height (above P) of the projectile as long as it is aloft. Note that $f(t) = 0$ when $t = 0$ and when $t = \frac{2v_0}{g}$. This means that the projectile returns to the initial 20 foot level after $\frac{2v_0}{g}$ seconds.

(a) Find the velocity of the projectile at $t = t_o$ (in terms of v_0 and g).

(b) Sketch the s vs. t and the v vs. t graphs on the same set of axes.

(c) Compute (in terms of v_0) the time required for the velocity to drop to zero.

(d) What is the velocity on return to the initial 20 foot level?

(e) Assume that the projectile returns to earth at a point 30 feet below the initial take off point P. What is the velocity at impact?

10. Show that if a ball is thrown upward with an initial velocity of V_0 ft./sec., it will reach a maximum height of $\frac{V_0^2}{64}$ feet.

11. Elsie can throw a ball 148 feet straight up. How fast does it go when it leaves her hand? (Assume that when the ball is released her hand is 4 ft. above the ground.)

12. A ball is thrown upward with an initial speed of 64 ft./sec. Simultaneously a ball is dropped from rest at a height of 128 ft. When does impact occur and how fast is each ball going at the time of impact?

13. Determine the average velocity of a car for a total trip if it averages 60 miles per hour going and 30 miles per hour returning.

14. Find the velocity of an object whose location along a straight line is described by the equation $s = 128t - 16t^2$. Sketch the graphs of s vs. t and v vs. t on the same set of axes.

(a) During what time interval or intervals is the object moving toward the location $s = 0$?

(b) What are the values of v and t when s is a maximum?

15. A ball is thrown upward with a velocity of 32 feet per second. Its height s in feet after t seconds is described by the equation $s = 32t - 16t^2$.

    (a) What is the velocity of the ball when its height first reaches 12 feet? When it again reaches 12 feet?

    (b) How high does it go, and how long after being thrown does the ball reach its highest position?

16. An object is projected up a smooth inclined plane in a straight line. Its distance s in feet from the starting point after t seconds is described by the equation $s = 64t - 8t^2$. After the object reaches its highest point it slides back along its original path to the starting point according to the equation $\bar{s} = 8(t - t_n)^2$. Here $\bar{s}$ is the distance of the object from the highest point and t_n is the time it takes the object to reach the highest point.

    (a) Determine how long it takes for the object to make the up and down trip.

    (b) Sketch the s vs. t graph for the up and down motion using one set of coordinates. Do the same for the v vs. t graph.

2-9. The Second Derivative

In the preceeding section, we found that the function expressing the acceleration of an object with respect to time is the derivative of its velocity function, which is in turn the derivative of the function f describing the object's position at time t. Hence, the acceleration function can be obtained by differentiating f twice. For any function f the derivative of f' is called the second derivative of f. The second derivative, denoted by f'', gives us valuable information about the graphs of both f' and f.

Consider the function

$$f : x \to x^3 - 3x.$$

Its first derivative is

$$f': x \to 3x^2 - 3,$$

and its second derivative is

$$f'': \to 6x.$$

Since f'' describes the slope of the tangent to the graph of f', and since $f''(x) = 6x < 0$, for $x < 0$, then f' is decreasing as $x < 0$ increases. Now to say that f', the derivative of f, is decreasing over an interval, is the same as saying that the slope of f is decreasing over that interval. From the sketch of Figure 2-9a we see that if the slopes of successive tangents to the graph of f are decreasing, the graph of f is "bending down." Hence, we are led to believe that the graph of the function

$$f : x \to x^3 - 3x$$

is similarly "bending down" as $x < 0$ increases, since $f''(x) < 0$ for $x < 0$.

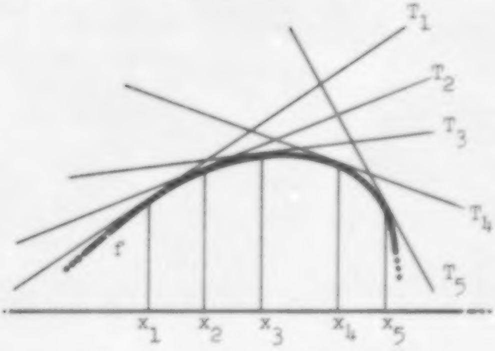


Figure 2-9a

In like manner, since $f''(x) = 6x > 0$ for $x > 0$, the slope of f, given by $f'(x)$, is increasing for $x > 0$. Figure 2-9b indicates that in such a situation the graph of f is "bending up." Putting these two pieces of information about the graph of $f : x \to x^3 - 3x$ together with what the first derivative, f', tells us about it, we obtain the following graph of f.

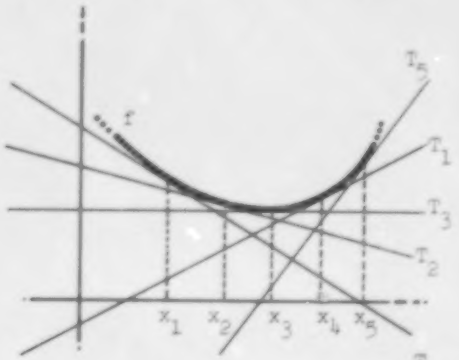


Figure 2-9b

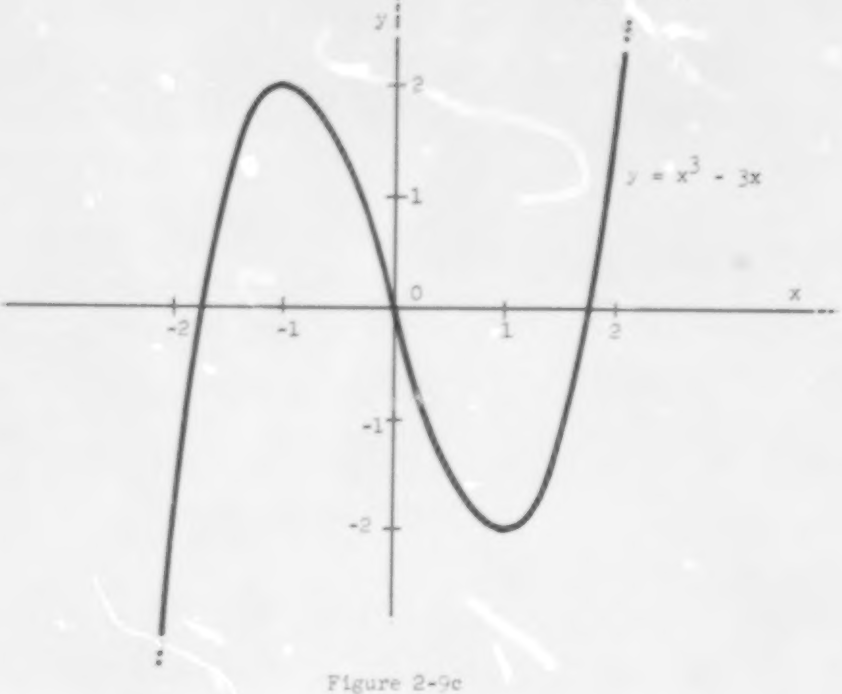


Figure 2-9c

The second derivative, $f''(x) = 6x$, indicates that for $x < 0$, the graph of f is bending downward, and for $x > 0$ the graph is bending upward. At $x = 0$, $f''(x) = 0$ and the graph is apparently changing from one to the other. The first derivative, $f' : x \to 3x^2 - 3 = 3(x - 1)(x + 1)$, indicates that the slope of the graph is zero at $x = \pm 1$ and from the way the graph is bending we conclude that f has a relative maximum at $x = -1$ and a relative minimum at $x = +1$.

A more rigorous interpretation of "bending upward" or "downward" uses the notion of the convexity and concavity of the function f. We say that a function is convex in the interval [a,b] if its graph in this interval lies above each of its tangents in the interval.[*] Figure 2-9b shows a function which is convex in the interval shown. Similarly a function is said to be concave in an interval [a,b] if its graph lies below each of its tangents in that interval. See Figure 2-9a for an example of a concave function.

Some texts use concave upward in place of convex and concave downward in place of concave. The ideas of convexity and concavity are said to describe the flexure (bending) of curves.

The intuitive remarks motivating the use of f'' in graphing $f : x \to x^3 - 3x$ lead to the characterizations that:

(1) a function f is convex in the interval [a,b] if and only if $f''(x) > 0$ for all x between a and b $(a < x < b)$.

and

(2) a function f is concave in the interval [a,b] if and only if $f''(x) < 0$ for all x between a and b $(a < x < b)$.

An important consequence of these two characterizations is that if the graph of a polynomial f crosses its tangent at the point of tangency $P(c,f(c))$, then $f''(c) = 0$.

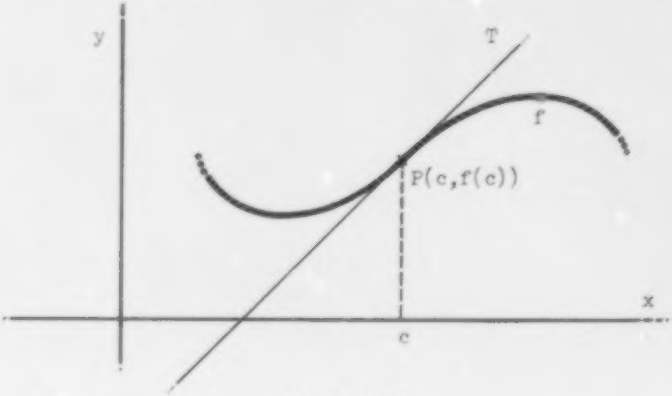


Figure 2-9d

The graph of f crosses its tangent at P.

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[*] The reader familiar with convex regions from a geometry course will observe that we are defining a function to be convex over the interval in which the region above the graph of f is convex.

For if the graph of f crosses T at $(c,f(c))$ then near $x = c$ (see Figure 2-9d), the graph must lie above the tangent on one side of c and below the tangent on the other side of c. Hence, $f''(x) > 0$ for x on one side of c and $f''(x) < 0$ on the other. Thus, $f''(c) = 0$ since a polynomial function must pass through zero as it passes from positive to negative.

> If the graph of f crosses its tangent at the point of tangency P, then P is called a *point of inflection* of f.

The above argument shows that if $P(c,f(c))$ is a point of inflection of f, then $f''(c) = 0$.

The converse of this may be false, however. It is quite possible that $f''(c) = 0$ at points $P(c,f(c))$ where the tangent does *not* cross the graph of f. Consider the graph of $f : x \to x^4$ at the origin. (See Exercises 2-9, No. 2.)

To summarize, it is instructive to view the graphs of f, f', and f'' together to see how the zeros of f' and f'' affect the graph of f. To show the relationships most vividly, we illustrate the graphs in Figure 2-9e without y-axes.

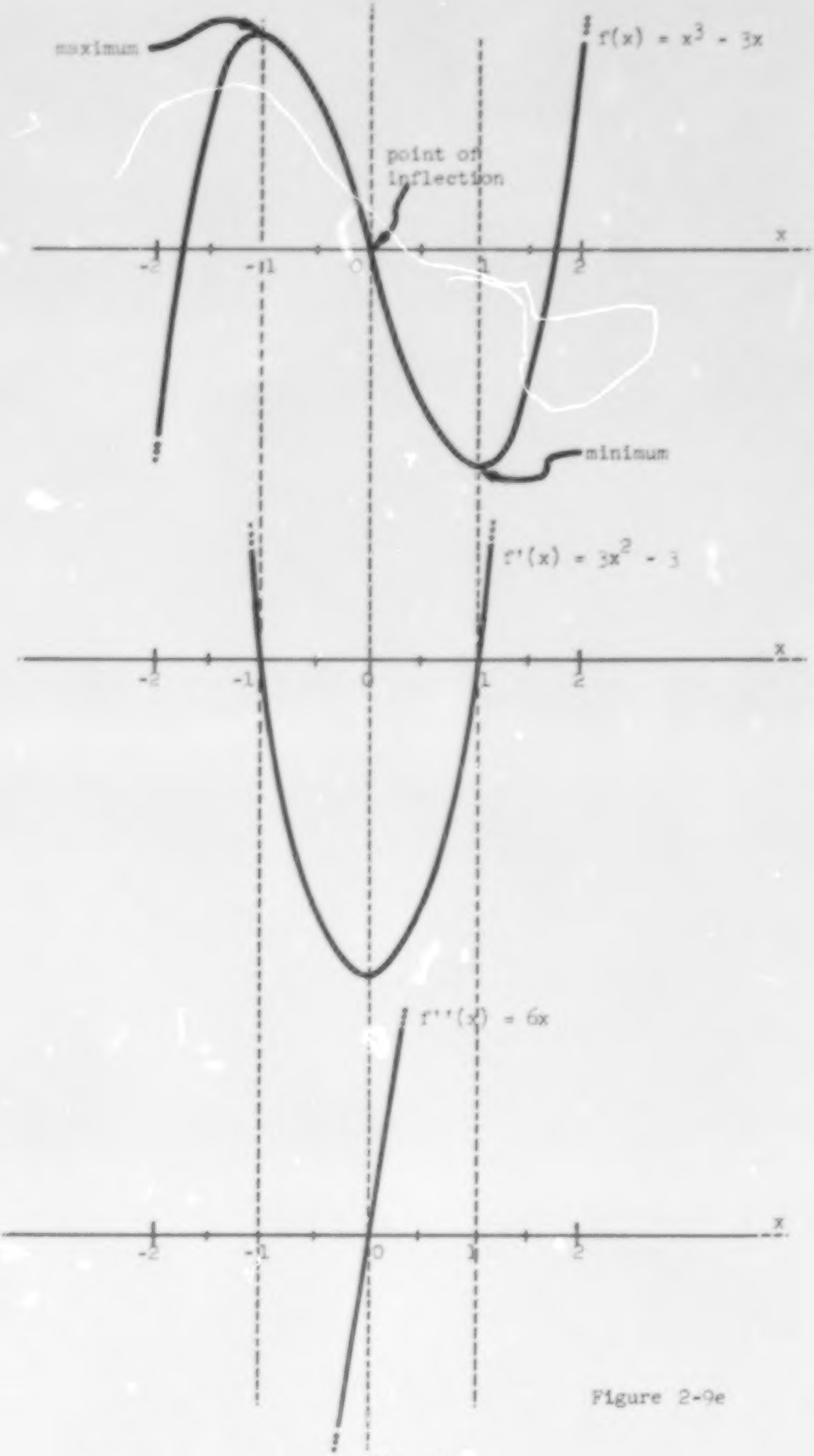

maximum
$f(x) = x^3 - 3x$
point of inflection
minimum
$f'(x) = 3x^2 - 3$
$f''(x) = 6x$
x
-2
-1
0
1
2

Figure 2-9e

Note that a relative maximum occurs at $(c,f(c))$ when $f'(c) = 0$ and $f''(c) < 0$, since the tangent to the graph of f at $x = c$ must be horizontal and the graph must be concave. Similarly, a relative minimum occurs when $f'(c) = 0$ and $f''(c) > 0$, since the tangent must be horizontal as before, but here the graph must be convex.

A point of inflection occurs at $(c,f(c))$ when $f''(c) = 0$ and $f''(x)$ is positive immediately to one side of $x = c$ and negative immediately to the other side of $x = c$.

Example 2-9a. Determine the intervals over which the function

$$f : x \to x^4 - 2x^2 + 1$$

is increasing, decreasing, convex, and concave, and locate all relative maxima and minima, and all points of inflection.

Intervals on which f is increasing and decreasing can be determined by the sign of the derivative

$$f': x \to 4x^3 - 4x = 4x(x + 1)(x - 1).$$

The critical points (points where $f'(x) = 0$) occur where $x = -1$, $x = 0$, and $x = 1$.

$$\begin{array}{lll} f'(x) < 0 & \text{for} & x < -1 \\ f'(x) > 0 & \text{for} & -1 < x < 0 \\ f'(x) < 0 & \text{for} & 0 < x < 1 \\ f'(x) > 0 & \text{for} & 1 < x. \end{array}$$

From these signs we conclude that as x increases the graph of f falls for $x < -1$, rises between -1 and 0, falls again between 0 and 1, and finally rises again for $x > 1$.

Intervals of convexity and concavity can be determined by the sign of the second derivative

$$f'': x \to 12x^2 - 4 = 4(3x^2 - 1) = 4(x\sqrt{3} - 1)(x\sqrt{3} + 1)$$

$f''(x) = 0$ if and only if $x = \sqrt{\frac{1}{3}}$ or $x = -\sqrt{\frac{1}{3}}$. $f''(x) < 0$ if and only if $(x\sqrt{3} - 1)$ and $(x\sqrt{3} + 1)$ have opposite signs, which happens if and only if $(x\sqrt{3} + 1) > 0$ and $(x\sqrt{3} - 1) < 0$, that is, if and only if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

Similarly, $f''(x) > 0$ if and only if $(x\sqrt{3} - 1)$ and $(x\sqrt{3} + 1)$ have the same sign which happens if and only if

$$x\sqrt{3} + 1 < 0 \quad \text{or} \quad x\sqrt{3} - 1 > 0,$$

i.e., if and only if $x < -\frac{1}{\sqrt{3}}$ or $x > \frac{1}{\sqrt{3}}$. We conclude that f is concave over the interval $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, and convex for all $x < -\frac{1}{\sqrt{3}}$ or all $x > \frac{1}{\sqrt{3}}$. The graph of f thus has points of inflection at $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$ since $f''\left(\pm\frac{1}{\sqrt{3}}\right) = 0$ and f is convex on one side of each of these points and concave on the other side. Together with the information that the first derivative of f is zero at $x = -1$, 0, and 1, these intervals of convexity and concavity show that

f has a relative minimum at $x = -1$,
f has a relative maximum at $x = 0$,
and f has a relative minimum at $x = 1$.

Finally, the graph of f has points of inflection at $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$ since $f\left(\pm\frac{1}{\sqrt{3}}\right) = 0$ and f is convex on one side of each of these points and concave on the other side.

In Figure 2-9f we sketch the graph of f using the foregoing information.

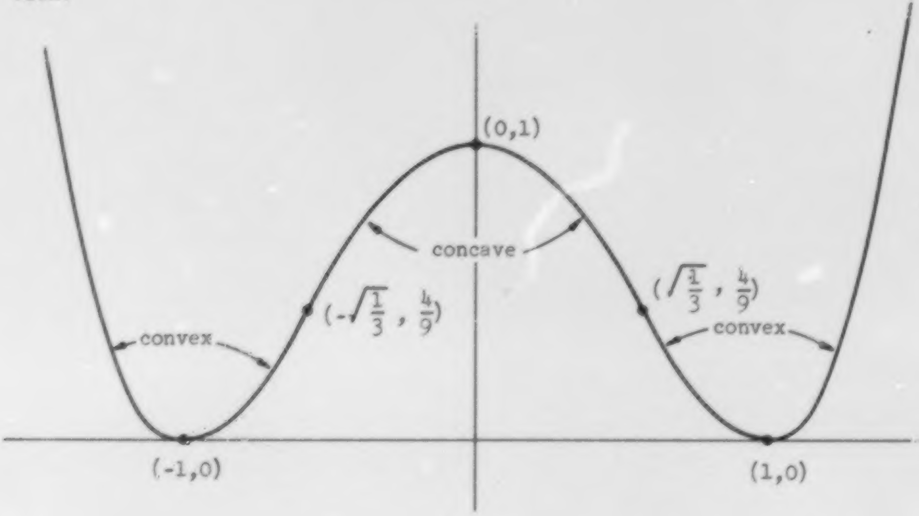


Figure 2-9f

$f : x \to x^4 - 2x^2 + 1$

Exercises 2-9

1. Determine the second derivative of the function
   $f : t \to 2t^3 - 39t^2 + 252t - 535$.

2. Characterize the origin for each of the following functions (by determining whether it is a relative max. or min., or point of inflection).
   (a) $f : x \to x^4$
   (b) $f : x \to x^4 - 4x^3$

3. Consider the function
   $$f : x \to 4x^5 + 5x^4 - 20x^3 - 50x^2 - 40x.$$
   (a) Find $f'(x)$ and $f''(x)$.
   (b) Characterize each of the points $(-1, f(-1))$ and $(2, f(2))$ as maximum or minimum.

4. Consider the function $f : x \to \frac{x^5}{5} - \frac{2x^3}{3} + x$ on $[-2,2]$.
   (a) Determine $f'(x)$.
   (b) Determine $f''(x)$.
   (c) Evaluate $f'(-1)$.
   (d) Evaluate $f''(-1)$.
   (e) Describe the behavior of f on $[-2,2]$ (by determining subintervals of increase, decrease, convexity, and concavity, and by locating relative maxima and minima and points of inflection, if any of these occur).
   (f) Sketch the graph of f on $[-2,2]$.

5. Determine the relative maximum point and relative minimum point of the graph of
   $$f : x \to (x + 2)^2(x - 2) .$$

6. Sketch the graph of
   $$f : x \to 4x^5 + 5x^4 - 20x^3 - 50x^2 - 40x.$$
   (See No. 3.)

7. Sketch the graph of $f : x \to 2x^3 - 3x^2 - 12x + 2$, indicating relative extrema (maxima and minima) and points of inflection.

8. The point $(1,1)$ lies on the graph of each of the following polynomial functions. For which is this point (i) a relative maximum, (ii) a relative minimum, (iii) a point of inflection, (iv) none of these?

   (a) $x \to 2x^3 - 6x^2 + 6x - 1$

   (b) $x \to 2x^3 - 6x + 5$

   (c) $x \to 2x^3 - 3x^2 + 12x - 10$

   (d) $x \to 2x^3 - 3x^2 - 12x + 14$

9. Consider the function $f : x \to x^4 + x^3 - 2x^2 - 3x$ over the interval $-2 \leq x \leq 2$.

   (a) At what points is a tangent to the graph of f horizontal?

   (b) What are the relative minimum points?

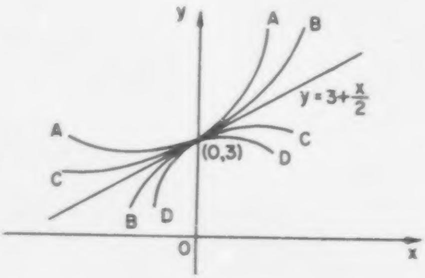
   (c) What is the minimum value of f?

   (d) What is the maximum value of f?

   (e) Sketch the graph of f.

10. Classify each of the points $(1,0)$, $(2,-2)$, and $(3,-4)$ on the graph of $x \to x^3 - 6x^2 + 9x - 4$ as a relative maximum, a relative minimum, a point of inflection, or none of these.

11. The figure at the left shows four polynomial graphs and their common tangent $y = 3 + \frac{x}{2}$ at $(0,3)$. Match each graph (A, B, C, D) with one of the following equations.



(a) $y = 3 - \frac{x}{2} - x^3$

(b) $y = 3 - \frac{x}{2} - x^2$

(c) $y = 3 - \frac{x}{2} + x^2$

(d) $y = 3 - \frac{x}{2} + x^3$

(e) $y = 3 + \frac{x}{2} - x^3$

(f) $y = 3 + \frac{x}{2} - x^2$

(g) $y = 3 + \frac{x}{2} + x^2$

(h) $y = 3 + \frac{x}{2} + x^3$

12. Classify (see No. 10) the point $(2,0)$ for each of the following functions.

(a) $x \to (x - 2)^2$

(b) $x \to (2 - x)^3$

(c) $x \to (x - 2)^4$

13. Find and classify each critical point (local extremum or point of inflection) for each of the following functions.

(a) $x \to 2x^3 + 3x^2 - 12x - 7$

(b) $x \to x^3 - 12x + 16$

(c) $x \to -2x^3 + 3x^2 + 12x + 7$

(d) $x \to (x - 1)^2(x + 2)$

14. Consider the function $f : x \to x^3 - 9x^2 + 24x - 18$.

(a) Determine f'.

(b) Locate the relative maximum and minimum points of f.

(c) Determine f''.

(d) What is the point of inflection of the graph of f?

(e) What is the slope of the tangent to the graph of f at $(3,0)$?

(f) Determine $f'(3 + k)$ and $f'(3 - k)$.

(g) Sketch the graph of f.

(h) Discuss the symmetry of the graph of f.

15. Show that the graph of a cubic function must have a point of inflection.

16. (a) Sketch the graph of

$$f : x \to x^5 + x^4 - 5x^3 - x^2 + 8x - 4.$$

Respond to each of the following by inspection of your graph for part (a).

(b) What are the zeros of f?

(c) Describe the flexure of the graph of f at the points where $f(x) = 0$.

(d) Describe the flexure of the graph of f at points for which $f'(x) = 0$ and $f(x) > -5$.

17. Determine those points on the graph of $f : x \to \frac{x^5}{5} - \frac{2x^2}{3} + x$ at which the flexure is neither upward nor downward.

18. Characterize the point $(\frac{5}{2}, -\frac{15}{16})$ on the graph of

$$f : x \to x^4 - 2x^3 - 7x^2 + 10x + 10.$$

19. Characterize the points $(0,0)$ and $(-1,-11)$ on the graph of $f : x \to 2x^6 + 3x^5 + 10x$.

20. Consider the functions $f : x \to (x + 1)^2(x - 2)$ and $g : x \to 3(x + 1)(x - 1)$.

    (a) What is the relationship between f and g?

    (b) Characterize each of the following points on the graphs of f and g.

    (i) $(-1,0)$
    (ii) $(0,f(0))$
    (iii) $(0,g(0))$
    (iv) $(1,g(1))$
    (v) $(1,f(1))$
    (vi) $(z,f(z))$

    (c) Sketch the graphs of f and g on the same set of axes.

21. Consider the function $f : x \to x^3 - 3x + 2$.

    (a) Locate the zeros of f.

    (b) Locate the relative maximum, relative minimum and point of inflection.

    (c) Sketch the graph.

22. Consider the function $f : x \to x^3 - 3x^2 + 4$.

    (a) Locate the zeros of f.

    (b) Locate the relative maximum, relative minimum and point of inflection.

    (c) Sketch the graph.

23. Show that the graph of $f : x \to Ax^2 + Bx + C$, $A \neq 0$, has no point of inflection.

24. Find an equation of the tangent to the graph of $f : x \to x^3 + 3x^2 - 4x - 3$ at its point of inflection.

2-10. Newton's Method

In Section 1-8 the method of repeated bisection was presented as a means for approximating zeros of a polynomial function. In this section we present another method, known as Newton's method for approximating such zeros. This method makes use of the derivative and is more efficient than repeated bisection.

Newton's method proceeds as follows. Suppose f is the given polynomial function and we wish to approximate the real zero r. By inspection of the graph of f, synthetic substitution, repeated bisection, or some other device, we obtain a first approximation of r. Let us call this first approximation x_1.

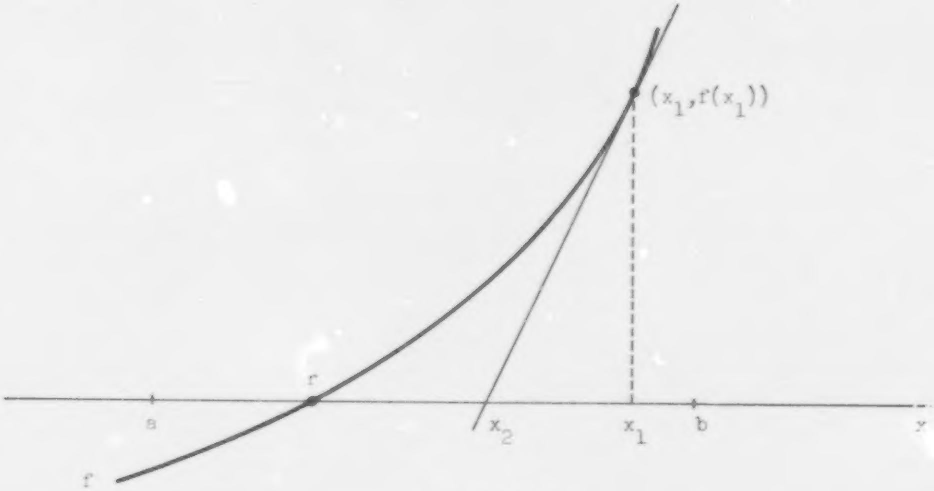


Figure 2-10a

If the graph of f looks like that shown in Figure 2-10a, we should expect that the tangent line at $(x_1,f(x_1))$ will intersect the x-axis at a point x_2, which is closer to r than is x_1. The tangent line at $(x_1,f(x_1))$ has the equation

$$y = f(x_1) + f'(x_1)(x - x_1).$$

This crosses the x-axis at $(x_2,0)$; that is,

$$0 = f(x_1) + f'(x_1)(x_2 - x_1).$$

Assuming that $f'(x_1) \neq 0$, we can solve for x_2, obtaining the formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} . \tag{1}$$

We can now repeat this process, using x_2 instead of x_1, to obtain the new approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} .$$

(See Figure 2-10b.)

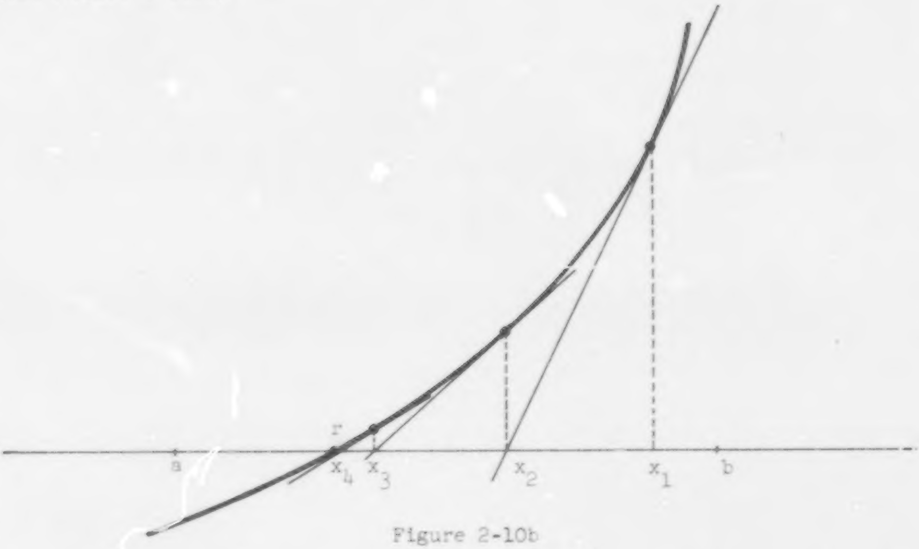


Figure 2-10b

Repeating again, using x_3 in place of x_1, we obtain the fourth approximation

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} .$$

Thus, equation (1) is the basis for an iterative* process; having arrived at the approximation x_n, we define a new approximation x_{n+1} by

(2) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} .$$

Example 2-10a. For the polynomial function

$$f : x \to x^3 + x^2 + x - 2$$

estimate the value of the real zero which lies between 0 and 1.

---

[*] A method of calculation which consists of the repetition (iteration) of a basic process, especially useful for writing a program for a computing machine.

Since $f(0) < 0$ and $f(1) > 0$, we know that there is (at least) one real zero between 0 and 1. Further calculation shows that

$$f(0.8) < 0 \text{ and } f(0.9) > 0,$$

so that the desired zero lies between 0.8 and 0.9. Let us take $x_1 = 0.8$ as our initial estimate. We have $f(0.8) = -0.048$. Since

$$f' : x \to 3x^2 + 2x + 1,$$

we have

$$f'(0.8) = 4.52;$$

whence formula (1) gives the second estimate

$$x_2 = 0.8 - (\frac{-.048}{4.52}) \approx 0.81.$$

Now we calculate to obtain

$$f(0.81) = -.002459$$

and

$$f'(0.81) = 4.5883.$$

We use (1), with x_1 replaced by x_2, to obtain the third estimate

$$x_3 = 0.81 - \frac{(-.002459)}{4.5883} \approx 0.8105.$$

Correct to two decimal places the zero of f is 0.81.

Example 2-10b. Use Newton's Method to estimate $\sqrt[3]{3}$.

Since $\sqrt[3]{3}$ is a root of the equation $x^3 = 3$, it must be a zero of the function

$$f : x \to x^3 - 3.$$

Since $f(1)$ and $f(2)$ have opposite signs we take $x_1 = 1.5$ as our first approximation. The derivative of f is

$$f' : x \to 3x^2$$

so that (1) gives

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{(1.5)^3 - 3}{3(1.5)^2}$$
$$\approx 1.444 \approx 1.44.$$

Using 1.44 as our second approximation, we obtain

$$x_3 = 1.44 - \frac{f(1.44)}{f'(1.44)} \approx 1.442.$$

Correct to two places $\sqrt[3]{3} \approx 1.44$.

Exercises 2-10

1. Use Example 2-10a to respond to each of the following.
   (a) What is the slope of the tangent to the graph of $f : x \rightarrow x^3 + x^2 + x - 2$ at the point $(0.8, -0.048)$?
   (b) Write the equation of the tangent to the graph of f at the point $(0.8, -0.048)$.
   (c) The line tangent to the graph of f at the point $(0.8, -0.048)$ intersects the x-axis at a point close to the place where the graph of f crosses the x-axis. What is the x-intercept of this tangent line?

2. Use Example 2-10b to respond to each of the following.
   (a) Find the slope of the tangent to the graph of $f : x \rightarrow x^3 - 3$ at the point $(1.5, 0.375)$.
   (b) What is the equation of the tangent to the graph of f at $(1.5, 0.375)$?
   (c) Find the value of x at which the tangent of part (b) intersects the x-axis.

3. (a) What is the positive zero of the function $f : x \rightarrow x^2 - 2$?
   (b) Show that a zero of f lies between 1 and 2.
   (c) Use Newton's method to approximate $\sqrt{2}$ to three decimal places.

4. Consider the function $f : x \rightarrow x^3 - 12x + 1$.
   (a) Show that there is at least one real number r such that $f(r) = 0$ and $0 < r < 1$.
   (b) Find f'.
   (c) Evaluate $f(0)$ and $f'(0)$.
   (d) To two decimal places approximate x_2 if $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ and $x_1 = 0$
   (e) Use your estimate from part (d) to show that

$$f(x_2) \approx 0.040512$$

and

$$f'(x_2) \approx -11.9808.$$

(f) Use Newton's Method and the results of parts (a) through (e) to compute the zero of f between 0 and 1 to three decimal places.

5. Calculate to two decimal places the zero of

$$f : x \to x^3 - 3x^2 + 2$$

which is between 2 and 3.

6. Find an approximate solution of

$$x^2 + 3x = 7$$

correct to two decimal places.

7. Suppose f is a polynomial function and $f(r) = 0$, $a < r < b$.

   (a) If the derivative $f'(x)$ changes sign over the interval $[a,b]$, it is possible that Newton's method will fail to generate closer and closer approximations to r. Sketch a picture showing such a situation.

   (b) If $f''(x)$ changes sign over the interval $[a,b]$, then even if $f'(x)$ does not change sign it is possible for Newton's method to fail. Sketch a picture showing such a situation.

   (c) In view of (a) and (b), what precautions should you take in applying Newton's method?

2-11. Higher Derivatives and Notation

We have denoted the derivative of the function f by the symbol f'. There are other notations in common use. In graphing f, we often write $y = f(x)$, so it is natural to write

$$y' = f'(x)$$

for the value of f' at x.

Another alternative symbol for $f'(x)$ is

$$Df(x).$$

This notation allows us to abbreviate such statements as

$$\text{if} \quad f : x \to ax^2 + bx + c$$

$$\text{then} \quad f'(x) = 2ax + b,$$

by writing

$$D(ax^2 + bx + c) = 2ax + b.$$

The symbol

$$\frac{dy}{dx}$$

which was introduced by Leibniz (1646 - 1716) to represent $f'(x)$, is suggested by the difference quotient used to calculate it. We have defined

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} .$$

If we replace h by the symbol "Δx" (read "delta x") to indicate a difference in x-coordinates, the difference quotient becomes

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} .$$

The expression $f(x + \Delta x) - f(x)$ stands for the corresponding difference in y coordinates, so we write

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x} .$$

This prompts the notation

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

for the value of the derivative if $y = f(x)$. The symbol $\frac{dy}{dx}$ is not a ratio;

it stands for the limit of a ratio. It is a tribute to the genius of Leibniz, however, that he chose a notation which, as we shall see, anticipates some properties of derivatives which permit us to handle their values as though they were common fractions.

Corresponding to the foregoing symbols for the first derivative, we have the following symbols for the value of the second derivative:

$$f''(x),\ y'',\ D^2f(x),\ \frac{d^2y}{dx^2}\ .$$

The Leibniz notation $\frac{d^2y}{dx^2}$ is again suggested by the difference quotient and the Δ symbol for "difference "

$$\frac{f'(x+\Delta x) - f'(x)}{\Delta x} = \frac{\Delta y'}{\Delta x} = \frac{\Delta(\frac{dy}{dx})}{\Delta x}\ .$$

To symbolize the limit as $\Delta x \to 0$, we write $\frac{d(\frac{dy}{dx})}{dx}$ or $\frac{d(dy)}{dx\ dx}$ in the shorthand $\frac{d^2y}{dx^2}$.

We have seen how to differentiate any polynomial function of the form

$$f : x \to a_0 + a_1x + a_2x^2 + \ldots + a_nx^n.$$

Since the second derivative of f is still a polynomial we may compute its derivative and call it the third derivative of f, denoting its value by any of the symbols

$$f'''(x),\ y''',\ D^3f(x),\ \frac{d^3y}{dx^3}\ .$$

Similarly, we could find the fourth derivative of f by differentiating the third derivative, and so on, to fifth, sixth, and even higher derivatives.

Geometrically, we have seen that $f'(x)$ can be interpreted as the slope of the tangent to the graph of f at the point $(x,f(x))$, and that the second derivative can be interpreted as an indicator of the curvature of the graph. Higher derivatives do not have such vivid geometric interpretations for f but they do have important algebraic relationships to the coefficients of the terms of f.

To be concrete let us look at a general third degree polynomial function

$$f : x \to a_0 + a_1 x + a_2 x^2 + a_3 x^3. \tag{1}$$

We observed in the first chapter that

$$f(0) = a_0.$$

If we obtain the derivative

$$f' : x \to a_1 + 2a_2 x + 3a_3 x^2,$$

we observe that

$$f'(0) = a_1.$$

We differentiate f' to obtain the second derivative

$$f'' : x \to 2a_2 + 2 \cdot 3a_3 x,$$

and observe that

$$f''(0) = 2a_2.$$

If we differentiate f'', we obtain the third derivative

$$f''' : x \to 2 \cdot 3a_3.$$

In this case

$$f'''(0) = \ \cdot 3a_3.$$

We summarize: for the cubic polynomial function

$$f : x \to a_0 + a_1 x + a_2 x^2 + a_3 x^3, \tag{1}$$

the coefficients are related to the values of f and its successive derivatives f', f'', and f''' at $x = 0$, by the formulas:

$$a_0 = f(0)$$
$$a_1 = f'(0)$$
$$a_2 = \frac{1}{2} f''(0)$$
$$a_3 = \frac{1}{2 \times 3} f'''(0).$$

Now we express f in terms of powers of $x - a$:

$$f : x \to b_0 + b_1(x - a) + b_2(x - a)^2 + b_3(x - a)^3. \tag{2}$$

Such an expression for f can be found by synthetic division as in Section 2-2. We can show that the coefficients b_0, b_1, b_2, and b_3 are given by

$$\begin{aligned} b_0 &= f(a) \\ b_1 &= f'(a) \\ b_2 &= \frac{1}{2} f''(a) \\ b_3 &= \frac{1}{2 \times 3} f'''(a). \end{aligned} \tag{3}$$

To show that $b_0 = f(a)$, we let $x = a$ in the expression for $f(x)$ to obtain

$$f(a) = b_0 + b_1(a - a) + b_2(a - a)^2 + b_3(a - a)^3 = b_0.$$

The remaining results (3) are almost as easy. We differentiate f to obtain the derivative

$$f' : x \to b_1 + 2b_2(x - a) + 3b_3(x - a)^2,$$

whence

$$f'(a) = b_1 + 2b_2(a - a) + 3b_3(a - a)^2 = b_1.$$

Differentiating f' we obtain

$$f'' : x \to 2b_2 + 2 \cdot 3b_3(x - a).$$

Therefore, we have

$$f''(a) = 2b_2, \quad \text{so that} \quad b_2 = \frac{1}{2} f''(a).$$

Another differentiation, gives

$$f''': x \to 2 \cdot 3b_3$$

so that

$$f'''(a) = 2 \cdot 3b_3 \quad \text{and hence} \quad b_3 = \frac{1}{2 \times 3} f'''(a).$$

Using the foregoing process for a fourth degree polynomial function

$$f : x \to b_0 + b_1(x - a) + b_2(x - a)^2 + b_3(x - a)^3 + b_4(x - a)^4$$

we could obtain

(4)
$$\begin{aligned} b_0 &= f(a) \\ b_1 &= f'(a) \\ b_2 &= \frac{1}{2} f''(a) \\ b_3 &= \frac{1}{2 \times 3} f'''(a) \\ b_4 &= \frac{1}{2 \times 3 \times 4} f''''(a) \end{aligned}$$

where f'''' is the fourth derivative of f; that is, f'''' is the derivative of f'''. It is common to write

$$f^{(4)}, \text{ rather than } f''''$$

for the fourth derivative of f; similarly we use the notation $f^{(5)}$, $f^{(6)}$, ..., for the fifth derivative, the sixth derivative, etc. It is also common to use the factorial notation

$$k! = 1 \times 2 \times 3 \times 4 \times \ldots \times k$$

with the convention that $0! = 1$.

Our results can be generalized: A polynomial function can be written as

$$f : x \to b_0 + b_1(x - a) + b_2(x - a)^2 + \ldots + b_n(x - a)^n,$$

where

(5) $b_0 = \frac{1}{0!} f(a)$, $b_1 = \frac{1}{1!} f'(a)$, $b_2 = \frac{1}{2!} f''(a)$, $b_3 = \frac{1}{3!} f'''(a)$,

and

(6) $$b_k = \frac{1}{k!} f^{(k)}(a), \quad k = 4, 5, \ldots, n.$$

Example 2-11a. Express $f : x \to 3 - 2x + 7x^4$ in terms of powers of $x + 1$. We have the successive derivatives:

$$f' : x \to -2 + 28x^3,$$
$$f'' : x \to 84x^2,$$
$$f''' : x \to 168x,$$
$$f^{(4)} : x \to 168.$$

Since $x + 1 = x - (-1)$, we need to find the values of these functions when $x = -1$; we have

$$f(-1) = 12; \quad b_0 = \frac{1}{0!} \times 12 = 12$$
$$f'(-1) = -30; \quad b_1 = \frac{1}{1!} \times (-30) = -30$$
$$f''(-1) = 84; \quad b_2 = \frac{1}{2!} \times 84 = 42$$
$$f'''(-1) = -168; \quad b_3 = \frac{1}{3!} \times (-168) = -28$$
$$f^{(4)}(-1) = 168; \quad b_4 = \frac{1}{4!} \times (168) = 7$$

Thus, we can write

$$f : x \to 12 - 30(x + 1) + 42(x + 1)^2 - 28(x + 1)^3 + 7(x + 1)^4.$$

The same result can, of course, be obtained by synthetic division. (See Exercises 2-11, No. 6.)

Exercises 2-11

1. How many nonzero derivatives can an n-th degree polynomial function have?

2. If we write a fifth degree polynomial function in the form
$f : x \rightarrow b_0 + b_1(x - a) + b_2(x - a)^2 + b_3(x - a)^3 + b_4(x - a)^4 + b_5(x - a)^5$,
then $b_5 = k \cdot f^{(5)}(a)$. What is the value of k?

3. We repeat part of Number 11 of Exercises 1-5 and again consider the function $f : x \rightarrow x^3 - 3x$. We submit a table to show three successive synthetic divisions of $f(x) = x^3 - 3x$ and resulting quotients by $x - 2$.

$$\begin{array}{rrrrl}
1 & 0 & -3 & 0 & \quad \underline{|2} \\
 & 2 & 4 & 2 & \\
\hline
1 & 2 & 1 & 2 & \\
\hline\hline
1 & 2 & 1 & & \quad \underline{|2} \\
 & 2 & 8 & & \\
\hline
1 & 4 & 9 & & \\
\hline\hline
1 & 4 & & & \quad \underline{|2} \\
 & 2 & & & \\
\hline
1 & 6 & & & \\
\hline\hline
\end{array}$$

(a) Determine $g(x)$ and $f(2)$ if

$$f(x) = (x - 2)g(x) + f(2).$$

(b) Determine $p(x)$ and $g(2)$ if

$$g(x) = (x - 2)p(x) + g(2).$$

(c) Determine $q(x)$ and $p(2)$ if

$$p(x) = (x - 2)q(x) + p(2).$$

(d) What is $q(2)$?

(e) Show that, for all x, we can write

$$f(x) = (x - 2)\{(x - 2)[(x - 2)q(2) + p(2)] + g(2)\} + f(2).$$

(f) Using the results of parts (a) through (e) of this problem determine A, B, C, and D if, for all x,

$$f(x) = x^3 - 3x = A(x - 2)^3 + B(x - 2)^2 + C(x - 2) + D.$$

(g) Find the first, second and third derivatives of $f : x \rightarrow x^3 - 3x$.

(h) Evaluate $f(2)$, $f'(2)$, $f''(2)$, and $f'''(2)$.

(i) Evaluate $\frac{f(2)}{0!}$, $\frac{f'(2)}{1!}$, $\frac{f''(2)}{2!}$, and $\frac{f'''(2)}{3!}$.

(j) Compare the results of parts (f) and (i).

4. Consider the functions

$$G : x \to x^3 - 3x$$

$$f : x \to 2$$

$$g : x \to 2 + 9(x - 2)$$

$$h : x \to 2 + 9(x - 2) + 6(x - 2)^2$$

$$F : x \to 2 + 9(x - 2) + 6(x - 2)^2 + 1(x - 2)^3.$$

(a) Find the value of each of these functions when $x = 2.1$.

(b) What quadratic function best represents the cubic function $G : x \to x^3 - 3x$ near the point where $x = 2$?

(c) What function serves as the best linear approximation to G near the point where $x = 2$?

(d) What function serves as the best quadratic approximation to G near the point where $x = -1$?

(e) What function serves as the best quadratic approximation to G near the point where $x = a$?

5. Find the first four derivatives of each of the following functions

(a) $F : x \to \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

(b) $f : x \to \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

(c) $g : x \to \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$

6. (a) Compile a table similar to Number 3 to indicate four successive synthetic divisions of $7x^4 - 2x + 3$ by $x + 1$.

(b) Use the table of part (a) to write $f : x \to 7x^4 - 2x + 3$ in terms of powers of $x + 1$. Compare your result with the result of Example 2-11a.

(c) Write the functions which give the best linear, quadratic and cubic approximations to the graph of $f : x \to 7x^4 - 2x + 3$ near the point $(-1,12)$.

(d) At the point $(-1,12)$ is the graph of f rising or falling? Is the graph of f flexed upward or downward near the point $(-1,12)$?

7. Show that the third derivative of the function

$$f : x \to ax^2 + bx + c, \quad a \neq 0$$

is the zero function.

Chapter 3

CIRCULAR FUNCTIONS

Unlike the polynomial functions we have considered in the first two chapters certain functions have the property that their function values repeat themselves in the same order at regular intervals over the domain. Functions having this property are called periodic. Included in this important class are the circular (trigonometric) functions.

The simplest periodic motion is that of a wheel rotating on its axle. Each complete turn of the wheel brings it back to the position it held at the beginning. After a point of the wheel traverses a certain distance in its path about the axle, it returns to its initial position and retraces its course again. The distance traversed by the point in a complete cycle of its motion is again a period, a period measured in units of length instead of units of time. If it should happen that equal lengths are traversed in equal times, the motion becomes periodic in time as well and the wheel can be used as a clock.

The model of a wheel rotating provides a basis for our definitions of the sine and cosine functions, whose values are defined as the second and first coordinates, respectively, of points on a circle of radius one. These definitions are compatible with those of ratios of sides of right triangles. By defining the sine and cosine functions in terms of a unit circle, their periodicity is immediately evident. Furthermore, we can use the geometric properties of circles to obtain the properties and graphs of these circular functions (Sections 3-1, 3-2, 3-3).

These definitions and results are applied to uniform circular motions (such as rotating wheels) in Section 3-4. The basic addition formulas are derived in Section 3-5, again by making use of the geometry of circles. These are applied in the next section to the study of pure waves, the simplest type of periodic motion, while the final section points toward some of the ways in which the circular functions can be used to analyze more general periodic phenomena.

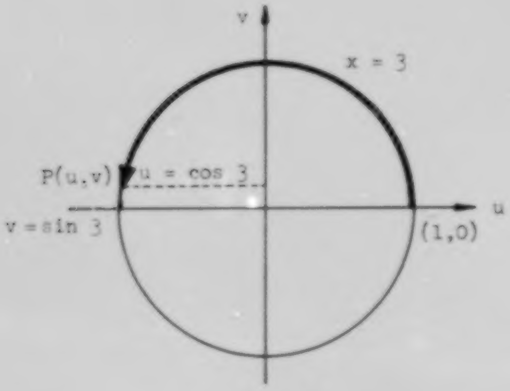
3-1. The Sine and Cosine Functions

We assume that you have some familiarity with the sine and cosine functions, so that much of the material in this chapter is review. You may have previously defined these functions in terms of ratios of sides of right triangles. We prefer, instead, to define the sine and cosine as functions of arc length on a circle. The usual angular definitions in terms of degree measurement can be obtained from our definitions by a suitable change of scale. Our definitions, in terms of the circle, have two great advantages: first, we can easily read off many properties of sine and cosine from properties of the circle; second and more important, our choice of scale will simplify our differentiation formulas.

For convenience of definition we use the circle with center at the origin and radius 1, the unit circle whose equation is

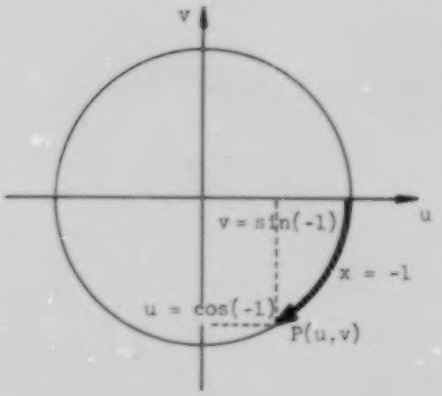
$$u^2 + v^2 = 1.$$

The circumference of the unit circle is 2π units. For any real number x we measure x units around this circle beginning at the point $(1,0)$. If x is positive we measure in a counterclockwise direction and if x is negative we measure in a clockwise direction. We obtain in this way a point P with coordinates (u,v) on the circle given by $u^2 + v^2 = 1$. The first coordinate of P is called the cosine of x, while the second coordinate of P is called the sine of x. (See Figure 3-1a and 3-1b.)



Note that $x > 0$ and we measure in a counterclockwise direction, obtaining $P(u,v)$ with $u = \cos 3$, $v = \sin 3$

Figure 3-1a



Note that $x < 0$, and we measure in a clockwise direction, obtaining $P(u,v)$ with $u = \cos(-1)$, $v = \sin(-1)$.

Figure 3-1b

Two functions, cosine and sine (abbreviated cos and sin), are defined as follows:

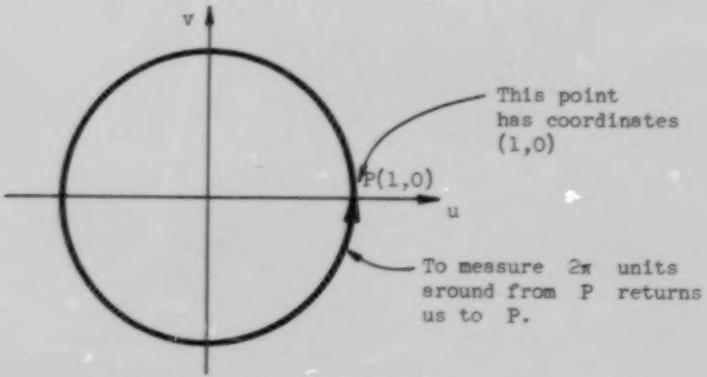
$$\boxed{\begin{aligned}\cos &: x \to u = \cos x = \text{the \underline{first} coordinate (abscissa) of } P\\ \sin &: x \to v = \sin x = \text{the \underline{second} coordinate (ordinate) of } P.\end{aligned}} \tag{1}$$

The values of cos and sin are easily obtained in certain cases. For example, referring to Figure 3-1c, we see that since P is the point $(1,0)$, we have, by definition

$$\cos 0 = 1 \quad \text{and} \quad \sin 0 = 0.$$

Since the unit circle has circumference 2π units we can measure 2π units around (in either direction) to obtain again the point P of Figure 3-1c. Thus

$$\begin{aligned}\cos 2\pi &= \cos(-2\pi) = 1\\ \sin 2\pi &= \sin(-2\pi) = 0.\end{aligned}$$



This point has coordinates (1,0)

To measure 2π units around from P returns us to P.

Figure 3-1c

To traverse one-fourth of the way around the unit circle is to move through $\frac{2\pi}{4} = \frac{\pi}{2}$ units. Thus if $x = \frac{\pi}{2}$ we have P_1 with coordinates $(0,1)$, so that

$$\cos\frac{\pi}{2} = 0, \quad \sin\frac{\pi}{2} = 1;$$

if $x = -\frac{\pi}{2}$ then we get P_2 with coordinates $(0,-1)$, so that

$$\cos(-\frac{\pi}{2}) = 0, \quad \sin(-\frac{\pi}{2}) = -1.$$

(See Figure 3-1d.)

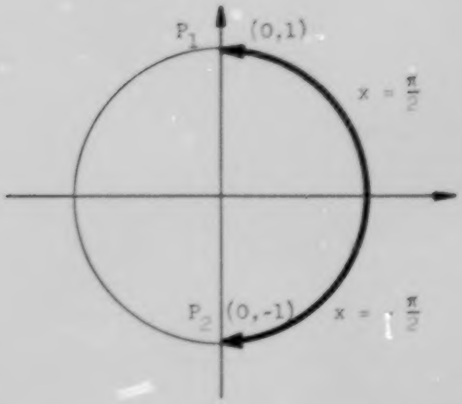


Figure 3-1d

Further calculations are indicated at the end of this section and in the exercises.

The sine and cosine are often defined in terms of ratios of sides of right triangles. In Figure 3-1e, the sine and cosine of angle AOB are defined by

$$\sin \angle AOB = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AB}{OA}$$

(2)

$$\cos \angle AOB = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{OB}{OA}$$

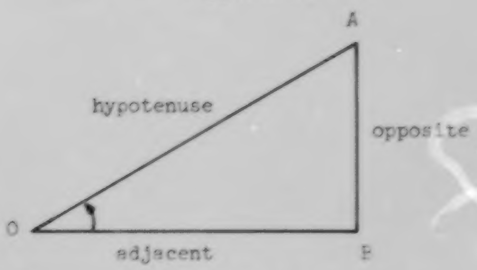


Figure 3-1e

To relate these definitions to our earlier ones, we can place the u and v axes as shown in Figure 3-1f, letting x denote the distance along the circle from $R(1,0)$ to P. The coordinates of P are $(\cos x, \sin x)$.

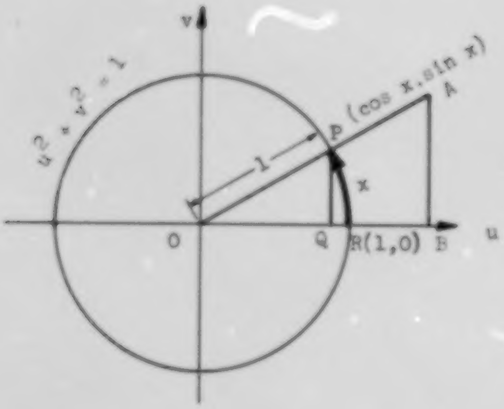


Figure 3-1f

Whether $OA > OP$ (as shown) or $OA < OP$, we have, by similar triangles that

$$\cos x = OQ = \frac{OB}{OA} = \cos \angle AOB$$

and

$$\sin x = PQ = \frac{AB}{OA} = \sin \angle AOB.$$

Thus the angle AOB corresponds to an arc of length x and $\cos x$ and $\sin x$ are respectively $\cos \angle AOB$ and $\sin \angle AOB$.

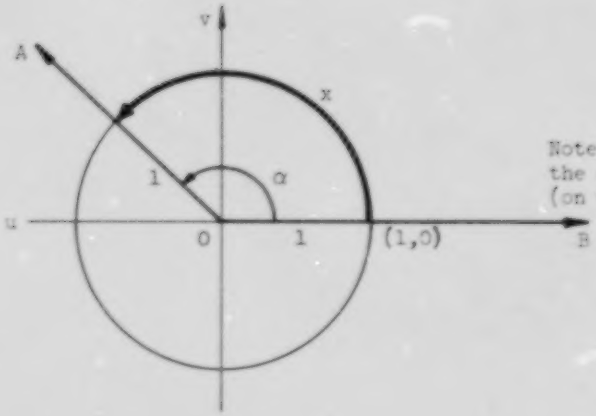
The right triangle definitions are somewhat restrictive as the angle AOB must always be between the zero angle and a right angle; that is, the arc length x must be between 0 and $\frac{\pi}{2}$. Our definitions (1) involve no such restriction and enable us to define $\sin x$ and $\cos x$ for any real number x. Thus (1) gives us a "natural" extension of the definitions (2).

Angular Measure

It is also common practice to measure angles in degrees. Degree measure is established by dividing the circle into 360 equal units, measuring an angle AOB by the number of units of arc it includes. For example, if $\angle AOB$ includes $\frac{1}{6}$ of the circumference we would say that the angle measures

$$\frac{1}{6} \times 360^\circ = 60^\circ$$

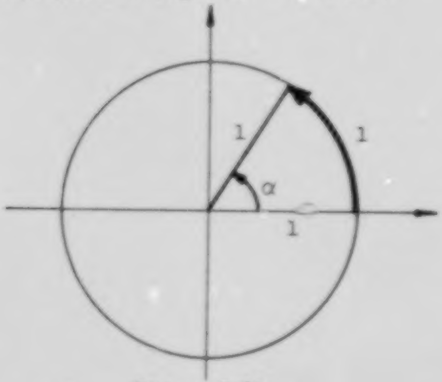
We can also measure angles by arc length.



Note that $\angle AOB$ determines the arc length x units (on the unit circle).

Figure 3-1g

In Figure 3-1g angle α cuts off an arc of length x on the circle given by $u^2 + v^2 = 1$. We say that α measures x units. This type of measure is called radian measure, its unit being called a radian. In Figure 3-1h we illustrate an angle of 1 radian.



α measures 1 radian.

Figure 3-1h

A moment's thought indicates the relationship between radian and degree measure. Clearly, if the radian measure of an angle is doubled, the degree measure also doubles. The same result is obviously true for halving, tripling, etc. In general, we have that the degree measure M of an angle α is directly proportional to the radian measure x. Thus

$$M = kx$$

where k is constant. Since $M = 360$ when $x = 2\pi$ we have

$$360 = k(2\pi)$$

so we must have $k = \frac{180}{\pi}$. Thus

(3) $\boxed{x \text{ radians corresponds to } \frac{180x}{\pi} \text{ degrees.}}$

We thus see that degree measure is obtained from radian measure by changing scale.

We note the following consequences of (3).

(4) 1 radian corresponds to $\frac{180}{\pi} \approx 57.296$ degrees

and

(5) 1 degree corresponds to $\frac{\pi}{180} \approx 0.01745$ radians.

In working with radian measure, it is customary simply to write, for example, $\frac{\pi}{2}$ when we mean $\frac{\pi}{2}$ radians. With degree measure we shall always use the degree symbol, such as 90°, 45°, etc.

Example 3-1a. Evaluate $\sin 990^\circ$.

We see that 990° corresponds to $\frac{\pi}{180} \times 990 = \frac{11}{2}\pi$ radians. We measure $\frac{11}{2}\pi$ units around the unit circle in a counterclockwise direction. If we write

$$\frac{11\pi}{2} = \frac{8\pi}{2} + \frac{3\pi}{2} = 2(2\pi) + \frac{3\pi}{2},$$

we indicate two times around the circle plus a $\frac{3}{4}$ turn, suggesting arrival at the point $(0,-1)$. (See Figure 3-1i) Thus $\sin 990^\circ = -1$.

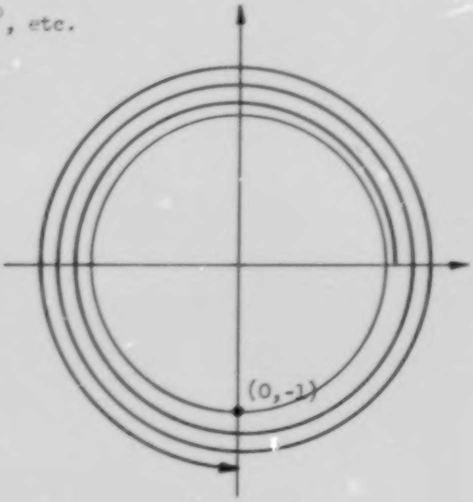


Figure 3-1i

Example 3-1b. If x is any multiple of $\frac{\pi}{3}$, determine $\cos x$ and $\sin x$.

Solution. The arclength $\frac{\pi}{3}$ is $\frac{1}{6}$th of a circle and hence corresponds to an angle of 60°. As Figure 3-1j shows we thus know that angles POQ and OQP are equal and hence that OR has length $\frac{1}{2}$ and PR has length $\frac{\sqrt{3}}{2}$.

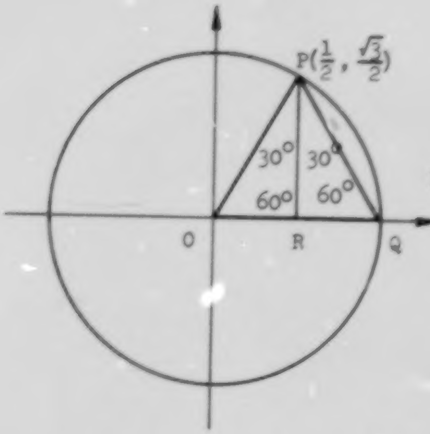

$P(\frac{1}{2}, \frac{\sqrt{3}}{2})$
30°
30°
60°
60°
O
R
Q

Figure 3-1j

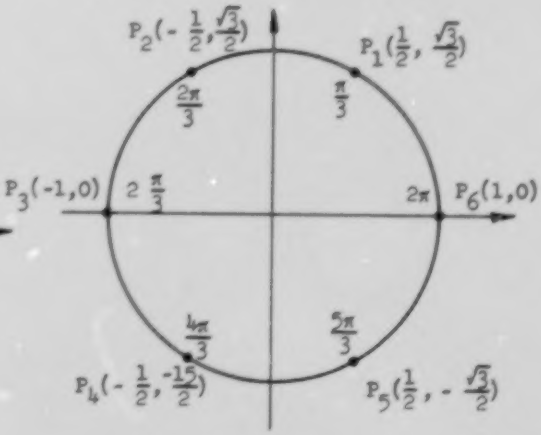

$P_2(-\frac{1}{2}, \frac{\sqrt{3}}{2})$
$P_1(\frac{1}{2}, \frac{\sqrt{3}}{2})$
$\frac{2\pi}{3}$
$\frac{\pi}{3}$
$P_3(-1,0)$
$2\frac{\pi}{3}$
2π
$P_6(1,0)$
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
$P_4(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$
$P_5(\frac{1}{2}, -\frac{\sqrt{3}}{2})$

Figure 3-1k

Thus P has coordinates

$$\cos\frac{\pi}{3} = \frac{1}{2} \quad \text{and} \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Measurements of $\frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$ give the respective points P_2, P_3, P_4, P_5, P_6 of Figure 3-1k. The coordinates of these points are easily found using the same techniques as above. This gives us enough information to find $\cos x$ and $\sin x$ for x any integer multiple of $\frac{\pi}{3}$. For example, if we wish to find $\cos(\frac{-16\pi}{3})$, we observe that

$$\frac{-16\pi}{3} = -5\pi - \frac{\pi}{3} .$$

We measure in a clockwise direction first 5π then $\frac{\pi}{3}$ units to obtain the point P_2. Thus $\cos(\frac{-16\pi}{3}) = -\frac{1}{2}$, the first coordinate (abscissa) of P_2.

Throughout our discussion we make use of the facts that

$$\begin{aligned} \sin(x - 2\pi) &= \sin x \\ \cos(x - 2\pi) &= \cos x \end{aligned} \tag{6}$$

The general form is stated as (5) in Section 3-2.

Exercises 3-1a

1. Change the following radian measure to degree measure.

   (a) $\frac{2\pi}{3}$

   (b) $\frac{\pi}{6}$

   (c) $\frac{-2\pi}{3}$

   (d) $\frac{7\pi}{6}$

   (e) 2π

   (f) $\frac{5\pi}{6}$

   (g) $\frac{8\pi}{3}$

   (h) $\frac{18\pi}{5}$

   (i) $\frac{13\pi}{4}$

   (j) 1.5

   (k) .7

   (ℓ) 2.6

2. Change the following degree measure to radian measure.

   (a) 270°

   (b) -30°

   (c) 135°

   (d) 480°

   (e) 195°

   (f) -105°

   (g) 810°

   (h) 190°

   (i) 18°

   (j) 0.4°

   (k) 1620°

   (ℓ) $\frac{18^\circ}{\pi}$

3. Express the following radian measure, in terms of the smallest positive angle:

   (a) What is the sum of the measures of the angles of a triangle? of a rectangle?

   (b) Given: a polygon of n sides. What is the sum of the measure of the interior angles? of the exterior angles?

   (c) The smaller of the two angles between the hands of a clock at 11:30 has a measure of __________.

   (d) Over which part of a radian does the minute hand of a clock move in 15 minutes? in 25 minutes?

   (e) How many radians does the minute hand sweep out in $1\frac{1}{2}$ hours? in 3 hrs. 50 min.?

4. Give the coordinates of the point on the unit circle corresponding to

(a) 300° (c) $\frac{22\pi}{3}$

(b) 1200° (d) 15π

5. Express each of the following angles in terms of a positive angle between 0 and 2π radians.

(a) $\frac{13}{6}\pi$ (c) $\frac{16}{7}\pi$

(b) $-\frac{4}{3}\pi$ (d) $-\frac{11}{12}\pi$

6. Write two equivalent expressions for each of the following angles in terms of

(i) $n(2\pi) + \alpha$ n integer, $|\alpha| < 2\pi$

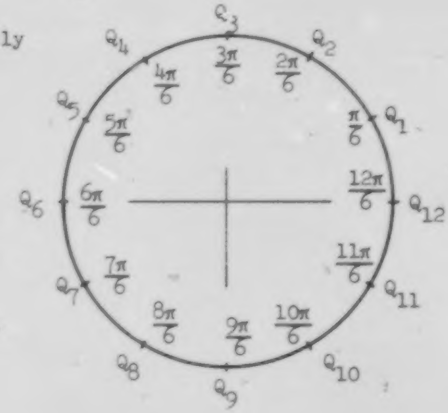
(ii) $n(\pi) + \alpha$ n integer, $|\alpha| < \pi$

(iii) $n(\frac{\pi}{2}) + \alpha$ n integer, $|\alpha| < \frac{\pi}{2}$

(a) $\frac{17}{4}\pi$ (c) $\frac{28\pi}{5}$

(b) $-\frac{31\pi}{12}$ (d) $-\frac{23\pi}{8}$

7. (a) Extending the information readily available from the $30\text{-}60\text{-}90^\circ$ triangle in Figure 3-1j, find $\cos x$ and $\sin x$ for x, a multiple of $\frac{\pi}{6}$, by drawing a unit circle similar to the one to the right and labelling the coordinates (cos x, sin x). For $Q_1, Q_2, \ldots, Q_{12}$ (similar to Figure 3-1k).



Q_3
Q_4
Q_2
$\frac{3\pi}{6}$
$\frac{4\pi}{6}$
$\frac{2\pi}{6}$
Q_5
$\frac{5\pi}{6}$
$\frac{\pi}{6}$
Q_1
Q_6
$\frac{6\pi}{6}$
$\frac{12\pi}{6}$
Q_{12}
$\frac{7\pi}{6}$
$\frac{11\pi}{6}$
Q_7
Q_{11}
$\frac{8\pi}{6}$
$\frac{9\pi}{6}$
$\frac{10\pi}{6}$
Q_8
Q_{10}
Q_9

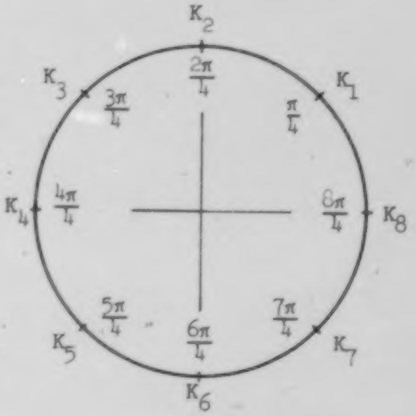
(b) Which of these points duplicate multiples of $\frac{\pi}{3}$ in Figure 3-1k?

(c) Which of the points $Q_1, Q_2, \ldots, Q_{12}$ have coordinates for the multiplies of $\frac{\pi}{2}$?

(d) Since $\frac{\pi}{4}$ is midway between $\frac{\pi}{6}$ and $\frac{\pi}{3}$, conjecture whether or not you can deduce $\cos\frac{\pi}{4}$ by averaging the value of $\cos\frac{\pi}{6}$ and $\cos\frac{\pi}{3}$? Can you make a general statement about reading off values of such functions from the drawing

$$\frac{\pi}{4} = \frac{\frac{\pi}{6}+\frac{\pi}{3}}{2} \quad \text{but} \quad \cos\frac{\pi}{4} \neq \frac{\cos\frac{\pi}{6}+\cos\frac{\pi}{3}}{2} \ ?$$

8. (a) Using relationships between the sides of a 45-45-90° triangle, find cos x and sin x for x, a multiple of $\frac{\pi}{4}$, by drawing a unit circle similar to the one to the right and labelling the coordinates (cos x, sin x) for $K_1, K_2, \ldots, K_8$.


K_2
$\frac{2\pi}{4}$
K_3
$\frac{3\pi}{4}$
K_1
$\frac{\pi}{4}$
K_4
$\frac{4\pi}{4}$
$\frac{8\pi}{4}$
K_8
$\frac{5\pi}{4}$
$\frac{6\pi}{4}$
$\frac{7\pi}{4}$
K_5
K_7
K_6

(b) From this circle read off the following values:

(1) $\sin\frac{3\pi}{4}$

(2) $\cos\frac{7\pi}{4}$

(3) $\sin\frac{13\pi}{4}$

(4) $\cos\frac{7\pi}{2}$

(5) $\sin(-\frac{5\pi}{4})$

(6) cos 135°

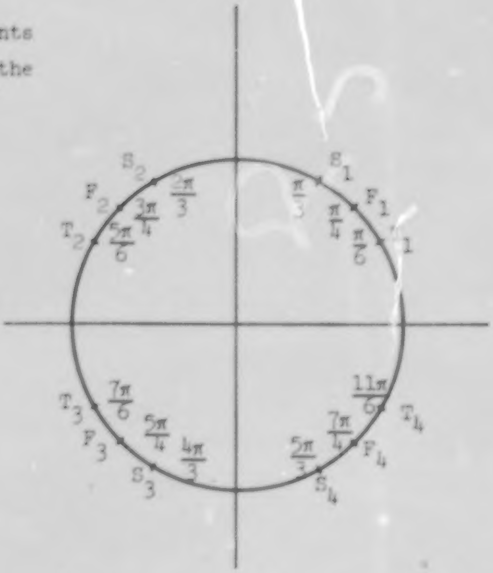
(7) sin 315°

(8) cos (-225°)

(9) sin (-135°)

(10) cos (3 · 360° + 45°)

9. Using the coordinates of the points indicated on the unit circle to the right answer the following:


S_1
F_1
T_1
S_2
F_2
T_2
T_3
F_3
S_3
T_4
F_4
S_4
$\frac{\pi}{3}$
$\frac{\pi}{4}$
$\frac{\pi}{6}$
$\frac{2\pi}{3}$
$\frac{3\pi}{4}$
$\frac{5\pi}{6}$
$\frac{7\pi}{6}$
$\frac{5\pi}{4}$
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
$\frac{7\pi}{4}$
$\frac{11\pi}{6}$

(a) Find the value of

$\sin\frac{\pi}{6}$, $\sin\frac{\pi}{4}$, $\sin\frac{\pi}{3}$;

$\cos\frac{\pi}{6}$, $\cos\frac{\pi}{4}$, $\cos\frac{\pi}{3}$.

What is the relationship between

$\sin\frac{\pi}{6}$ and $\cos\frac{\pi}{3}$?

$\sin\frac{\pi}{3}$ and $\cos\frac{\pi}{6}$?

$\sin\frac{\pi}{4}$ and $\cos\frac{\pi}{4}$?

In this first quadrant, what is the sign of the sine? of the cosine?

(b) Find the value of $\sin\frac{5\pi}{6}$, of $\sin\frac{3\pi}{4}$, and of $\sin\frac{2\pi}{3}$;

of $\cos\frac{5\pi}{6}$, of $\cos\frac{3\pi}{4}$, and of $\cos\frac{2\pi}{3}$.

In this second quadrant, what is the sign of the sine? of the cosine?

(c) Find the value of $\sin\frac{7\pi}{6}$, of $\sin\frac{5\pi}{4}$, and of $\sin\frac{4\pi}{3}$;

of $\cos\frac{7\pi}{6}$, of $\cos\frac{5\pi}{4}$, and of $\cos\frac{4\pi}{3}$.

In this third quadrant, what is the sign of the sine? of the cosine?

(d) Find the value of $\sin\frac{11\pi}{6}$, of $\sin\frac{7\pi}{4}$, and of $\sin\frac{5\pi}{3}$;

of $\cos\frac{11\pi}{6}$, of $\cos\frac{7\pi}{4}$, and of $\cos\frac{5\pi}{3}$.

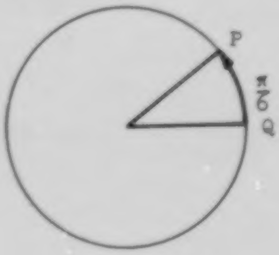
In this fourth quadrant, what is the sign of the sine? of the cosine?

(e) (i) In which quadrants is sine positive? sine negative?

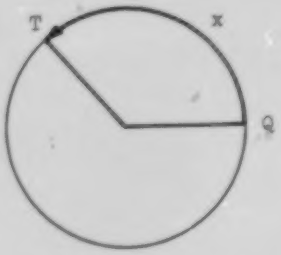
(ii) In which quadrants is cosine positive? cosine negative?

(iii) In which quadrant is sine positive and cosine negative? sine negative and cosine positive? both positive? both negative?

10. (a) What are the coordinates of P, indicated on the circle to the right, if the circle has a radius of 1? 2? $4\sqrt{3}$? R?



(b) What are the coordinates of T, indicated on the circle to the right, if the arc measure is x and the radius is 2? 7? $\frac{3}{2}$? r?



11. Given a circle of radius 1:

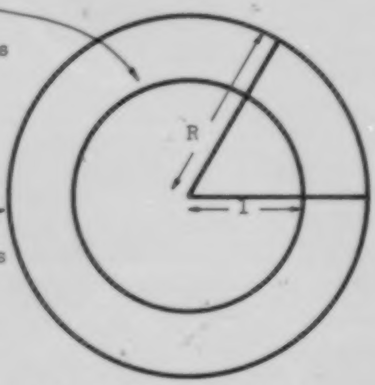
An arc which measures 1 radian has length 1;

an arc which measures x radians has length x.

Given a circle of radius R:

An arc which measures 1 radian has length R;

an arc which measures x radians has length xR.

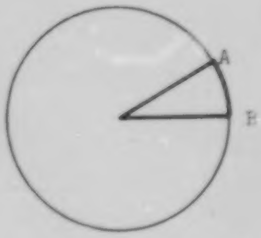


(a) Show by similar triangles that the length of the arc is proportional to the measure of the arc, and the constant of proportionality is the radius, or $\boxed{s = Rx}$

(b) The minute hand of a clock is 4 inches long. Approximately how far does its tip travel in 15 minutes?

(c) A circle has a radius of 15 inches. How long is the arc which measures 60°? 72°?

(d) What is the radius of the circle to the right if the measure of $\widehat{AB}$ is $\frac{\pi}{6}$, and the length of $\widehat{AB}$ is



(i) $\frac{\pi}{6}$ in.

(ii) $\frac{\pi}{3}$ in.

(iii) 10 in.

(iv) x in.

(v) $3x$ in.

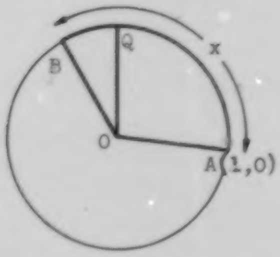
(e) What is the radius of a circle if the measure of $\widehat{AB}$ is x, and the length of $\widehat{AB}$ is x? $2x$? $10x$?

(f) If an arc of length π has a measure of $\frac{\pi}{10}$, what is the length of an arc of a semi-circle? of one-third of the circumference?

12. From geometry we know that in any circle the areas of two sectors of a circle are proportional to the measures of their arcs; for example:

$$\frac{\text{Area sector AOB}}{\text{Area sector AOQ}} = \frac{x}{\frac{\pi}{2}} ;$$

i.e., Area sector AOB $= \frac{2x}{\pi} \cdot \frac{\pi R^2}{4} = \frac{R^2 x}{2}$.



(a) This formula can be re-stated in this way:

In a given circle the area of a sector AOB = constant × arc measure $\widehat{AB}$ or $A = kx$, where the constant of proportionality, k, is $\frac{R^2}{2}$; that is,

$$\boxed{A = \frac{1}{2} R^2 x}$$

What is this constant of proportionality for a circle of radius 1? 2? 4?

(b) In a circle of radius a, if a given arc has a measure m, what is the area of the sector?

(c) In a given circle, how is the area of a sector affected if the arc measure is doubled? halved? tripled?

(d) In the beginning of this problem we stated that the area of sector AOB is proportional to the arc measure. Obviously, the area of sector AOB is also proportional to the arc length. What is this constant of proportionality?

(e) What is the area of a sector of a circle of radius 18 inches if the arc of the sector is 12 inches long?

(f) How is the area of the sector affected if the arc length is doubled? halved? tripled?

13. (a) Illustrate (6) geometrically; that is, show

$$\sin (x - 2\pi) = \sin x$$
$$\cos (x - 2\pi) = \cos x.$$

(b) Show that

$$\sin (x + 2n\pi) = \sin x$$
$$\cos (x + 2n\pi) = \cos x$$

where n is an integer.

The Use of Tables

In a separate booklet we give tables of approximate values of $\cos x$ and $\sin x$ for decimal values of x up to 1.60 which is slightly more than $\frac{\pi}{2}$. (The number x, of course, represents the measure of an arc length on the unit circle, i.e., radian measure.) More complete tables, tables in terms of degree measure and tables for converting from radian to degree measure are also found in the Booklet of Tables.

The following examples indicate some of the ways of using these tables:

Example 3-1c. Find $\sin .73$ and $\cos .73$.

We simply read from the tables the values

$$\sin .73 \approx .6669$$

and

$$\cos .73 \approx .7452.$$

Example 3-1d. Find $\sin 6.97$ and $\cos 6.97$.

While our tables do not include 6.97, we do know that

$$\sin x = \sin(x - 2\pi) \quad \text{and} \quad \cos x = \cos(x - 2\pi).$$

Using $2\pi \approx 6.28$, we have

$$\sin 6.97 \approx \sin .69 \approx .6365$$

and

$$\cos 6.97 \approx \cos .69 \approx .7712.$$

Example 3-1e. Find $\sin \frac{\pi}{6}$.

Using $\pi \approx 3.142$ we have

$$\frac{\pi}{6} \approx .524.$$

The tables give

$$\sin .52 \approx .4969$$

$$\sin .53 \approx .5055$$

Interpolating, we obtain

$$\sin .524 \approx .4969 + \frac{4}{10}(.5055 - .4969).$$

Therefore, [illegible] $\frac{\pi}{6} \approx .5003$.

Of course, we can observe that $\frac{\pi}{6}$ (radians) corresponds to 30° and read the result $\sin \frac{\pi}{6} = \frac{1}{2}$ from Figure 3-1ℓ.

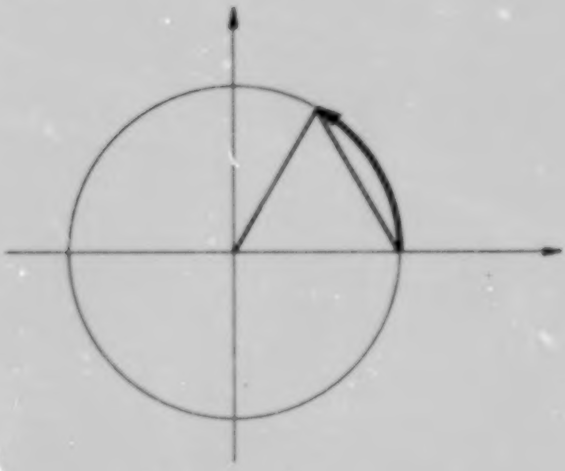


Figure 3-1ℓ

Example 3-1f. Find x such that $0 \le x \le \frac{\pi}{2}$ and $\sin x \approx .8850$.

From the tables we see that

$$\sin 1.08 \approx .8820$$

and

$$\sin 1.09 \approx .8866.$$

Interpolating we get

$$x \approx 1.08 + \frac{30}{46}(.01) \approx 1.0865 \approx 1.087.$$

Example 3-1g. Find $\sin 2$.

Referring to Figure 3-1m we see that

$$\begin{aligned} \sin 2 &= \sin(\pi - 2) \\ &\approx \sin(3.14 - 2) \\ &\approx \sin 1.14 \\ &\approx .9086 \end{aligned}$$

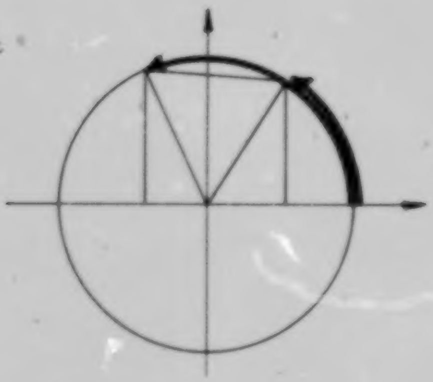


Figure 3-1m

Example 3-1h. Find $\sin 100$.

To find where the $\sin 100$ is located on the unit circle we ask how many times 2π divides 100. Since $2\pi \approx 6.283$ we guess that $\frac{100}{2\pi} \approx 16$. In fact $16 \times 2\pi \approx 16 \times 6.283 \approx 100.528$, so that $100 = 16 \times 2\pi - .53$. We show (Figure 3-1n) that point P is 100 units around the unit circle (counterclockwise) from (1,0) or .53 units short of 16 revolutions. The tables give

$$\sin .53 \approx .5055$$

so that

$$\sin 100 \approx -\sin .53 \approx -.5055.$$

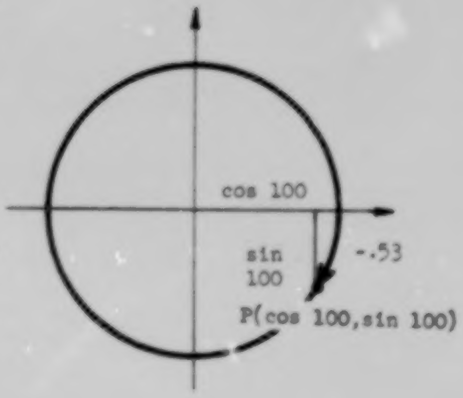


Figure 3-1n

Example 3-1i. Find $\cos 2000^\circ$.

Since there are 360° in one revolution we write $2000 = 5 \times 360 + \frac{1}{2} \times 360 + 20$. Five and one-half counterclockwise revolutions plus 20° gives a point on the unit circle 20° into the third quadrant. We have $\cos 2000^\circ = \cos 200^\circ = \cos(180^\circ + 20^\circ) = -\cos 20^\circ$. We use the table of No. 5 of Exercises 3-1b to find that $-\cos 20^\circ \approx -.940$. To use our radian tables we first note that 20° corresponds to .35 (approximately) so that

$$\cos 2000^\circ \approx -0.9394.$$

Exercises 3-1b

For problems 1, 2, 3, 4 use Table 3 in the Booklet of Tables

1. Find $\sin x$ and $\cos x$ when x is equal to

   (a) 0.73 (c) 1.55

   (b) -5.17 (d) 6.97 (Hint: $2\pi \approx 6.28$)

2. Find x when $0 \le x \le \frac{\pi}{2}$ and

   (a) $\sin x \approx 0.1098$ (c) $\sin x \approx 0.6518$

   (b) $\cos x \approx 0.9131$ (d) $\cos x \approx 0.5403$

3. Using $\pi \approx 3.14$, approximate the following, interpolating where necessary.

   (a) $\sin \frac{11}{12}\pi$ (c) $\sin 11.5$

   (b) $\cos \frac{7\pi}{5}$ (d) $\cos 417^\circ$

4. Find x where $0 \le x \le 1.57$

   (a) $\sin x = 0.2231$ (c) $\sin x = .8714$

   (b) $\cos x = 0.7135$ (d) $\cos x = .1759$

5. Below is a table giving values of $\sin x$ and $\cos x$ when x is given in degrees. $\sin x^\circ$ and $\cos x^\circ$ for angles between 0° and 45° are read from the top and left, $\sin x^\circ$ and $\cos x^\circ$ for angles between 45° and 90° are read from the bottom and right. For example, $\sin 20^\circ = \cos 70^\circ \approx 0.342$.

x°	$\sin^\circ x$	$\cos^\circ x$	
0°	0.000	1.000	90°
5°	0.087	0.996	85°
10°	0.174	0.985	80°
15°	0.259	0.966	75°
20°	0.342	0.940	70°
25°	0.423	0.906	65°
30°	0.500	0.866	60°
35°	0.574	0.819	55°
40°	0.643	0.766	50°
45°	0.707	0.707	45°
	$\cos^\circ x$	$\sin^\circ x$	x°

Using the table above find the values of the following:

(a) $\sin 75^\circ$

(b) $\cos 140^\circ$

(c) $\sin 480^\circ$

(d) $\cos(-460^\circ)$

6. Using the table in Number 5, find two values for x in degrees $0^\circ \leq x^\circ < 360^\circ$.

(a) $\sin x = 0.574$

(b) $\cos x = 0.643$

(c) $\sin x = -0.819$

(d) $\cos x = -0.087$

3-2. Properties of the Circular Functions

We have defined the circular functions, cosine and sine, by measuring arc length along the unit circle $u^2 + v^2 = 1$. Many properties of these two functions are easily derived from this definition. In this section we derive a few of these properties.

The values $\cos x$ and $\sin x$ were defined as the coordinates $(\cos x, \sin x)$ of a point P on the circle $u^2 + v^2 = 1$ such as in Figure 3-2a. Therefore, the coordinates of P must satisfy this equation, that is:

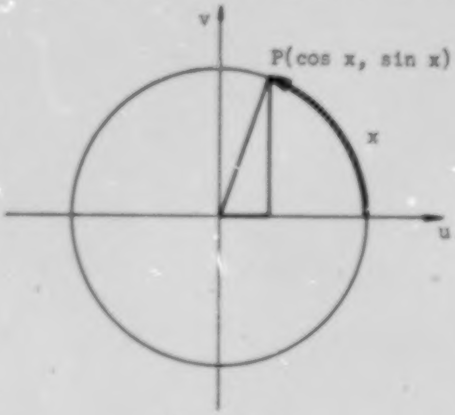


Figure 3-2a

(1) $$\boxed{\cos^2 x + \sin^2 x = 1}$$

This identity will often be useful. We have followed the usual convention of writing $\cos^2 x$ rather than $(\cos x)^2$, $\sin^2 x$ rather than $(\sin x)^2$.

Since a square is never negative it follows that

$$\cos^2 x \leq \cos^2 x + \sin^2 x,$$

and

$$\sin^2 x \leq \cos^2 x + \sin^2 x.$$

Combining these with (1) gives the two inequalities

$$\cos^2 x \leq 1 \text{ and } \sin^2 x \leq 1,$$

which can be rewritten as

(2) $$-1 \leq \cos x \leq 1 \text{ and } -1 \leq \sin x \leq 1.$$

Another consequence of (1) which will be useful in our approximation discussions in the next chapter is the inequality

(3) $$\boxed{0 \leq 1 - \cos x \leq \frac{x^2}{2} .}$$

To establish (3) we use the familiar distance formula to get (in Figure 3-2b) the distance from P to Q:

$$\sqrt{(1 - \cos x)^2 + \sin^2 x}$$

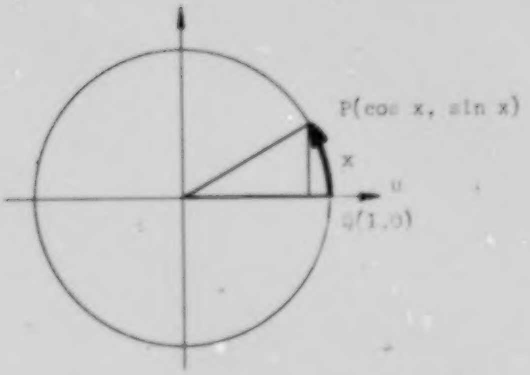


Figure 3-2b

This distance cannot exceed $|x|$, since the shortest distance between two points is measured along the straight line joining them. Thus

$$\sqrt{(1 - \cos x)^2 + \sin^2 x} \leq |x|.$$

Squaring and then multiplying out $(1 - \cos x)^2$ gives:

$$1 - 2 \cos x + \cos^2 x + \sin^2 x \leq x^2,$$

and hence (1) gives:

$$2 - 2 \cos x \leq x^2.$$

Dividing by 2, we get $1 - \cos x \leq \frac{x^2}{2}$. Noting that $\cos x \leq 1$ and hence $0 \leq 1 - \cos x$, we complete the proof of

(3) $$0 \leq 1 - \cos x \leq \frac{x^2}{2}.$$

Periodicity and Related Results

There are several formulas which relate the values $\sin x$ and $\cos x$ at different points. For example, if we traverse the unit circle 2π units, we arrive at our initial position, since the circle $u^2 + v^2 = 1$ has circumference 2π. (See Figure 3-2c) Thus we have

$$\sin (x + 2\pi) = \sin x \qquad (4)$$
$$\cos (x + 2\pi) = \cos x$$

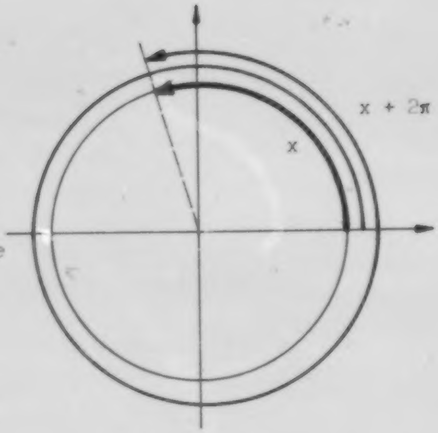


Figure 3-2c

Functions which repeat their values at equal intervals are said to be periodic. In general, if there is a number $a > 0$ such that

$$f(x + a) = f(x) \quad \text{for all } x,$$

then we say that f is periodic with period a. Thus the functions sin and cos are periodic with period 2π. As consequences of (4) we have

$$\begin{aligned} \sin (x + 4\pi) &= \sin \big((x + 2\pi) + 2\pi\big) \\ &= \sin (x + 2\pi) \\ &= \sin x \end{aligned}$$

and

$$\begin{aligned} \sin (x - 2\pi) &= \sin \big((x - 2\pi) + 2\pi\big) \\ &= \sin x \end{aligned}$$

In fact, for any integer n we can make the general statements

$$(5) \qquad \boxed{\begin{aligned} \sin (x + 2n\pi) &= \sin x \\ \cos (x + 2n\pi) &= \cos x. \end{aligned}}$$

Other useful formulas can be "read off" from the properties of the unit circle given by $u^2 + v^2 = 1$. For example, the points (u,v) and $(u,-v)$ are symmetric with respect to the u-axis. Consequently, we see (Figure 3-2d) that

$$(6) \qquad \boxed{\begin{aligned} \cos (-x) &= \cos x \\ \sin (-x) &= -\sin x \end{aligned}}$$

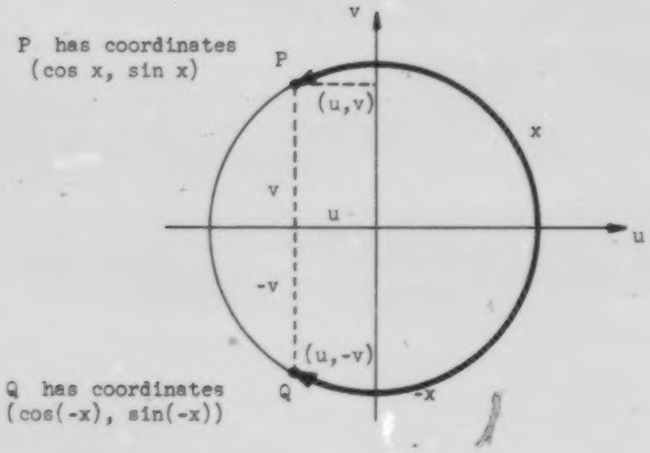

v
P has coordinates
(cos x, sin x)
P
(u,v)
x
v
u
u
-v
(u,-v)
Q has coordinates
(cos(-x), sin(-x))
Q
-x

Figure 3-2d

Using the unit circle we can also derive the two familiar formulas:

$$(7)\qquad \boxed{\begin{aligned}\sin\left(\tfrac{\pi}{2} - x\right) &= \cos x\\ \cos\left(\tfrac{\pi}{2} - x\right) &= \sin x\end{aligned}}$$

In Figure 3-2e triangle OPR is congruent to triangle OQS. (Why?) Then $P(u,v)$ and $Q(u_1,v_1)$ are related so that $u = v_1$ and $v = u_1$. It follows that

$$\cos x = OR = QS = \sin\left(\frac{\pi}{2} - x\right)$$

and

$$\sin x = PR = OS = \cos\left(\frac{\pi}{2} - x\right).$$

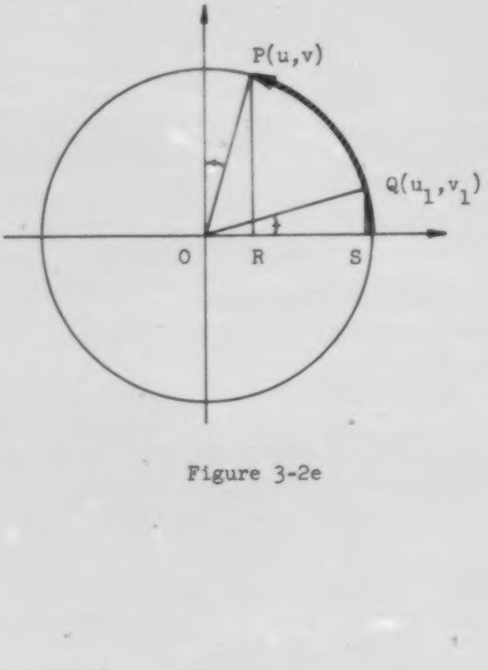

P(u,v)
Q(u₁,v₁)
O
R
S

Figure 3-2e

The useful formula

$$\cos x = \sin\left(x + \frac{\pi}{2}\right) \tag{8}$$

can also be derived by geometric arguments using the unit circle. Here we derive it using (6) and (7), as follows

$$\begin{aligned}\cos x = \cos(-x) &= \sin\left[\frac{\pi}{2} - (-x)\right]\\ &= \sin\left(x + \frac{\pi}{2}\right).\end{aligned}$$

We have given but a sample of the relationships which can be derived from the unit circle. Other such results will be derived in the exercises and, as we need them, in Sections 3-5 and 4-1.

Exercises 3-2

1. Using $f(x + 2n\pi) = f(x)$, and $f : x \rightarrow \cos x$, find

   (a) $f(3\pi)$ (d) $f(\frac{25\pi}{6})$

   (b) $f(\frac{7\pi}{3})$ (e) $f(-7\pi)$

   (c) $f(\frac{9\pi}{2})$ (f) $f(-\frac{10\pi}{3})$

2. If $f : x \rightarrow \sin x$, find the values of f in Exercise 1 above.

3. For what values of x (if any) will

   (a) $\sin x = \cos x$?

   (b) $\sin x = -\cos x$?

   (c) $\sin x = \sin(-x)$?

   (d) $\cos x = \cos(-x)$?

   Hint: Use the fact that $(\cos x, \sin x)$ represents a point on the unit circle.

4. (a) Using only the definition that $\sec\theta$ and $\csc\theta$ are reciprocals of $\cos\theta$ and $\sin\theta$, respectively, show that the expression $\frac{\sec\theta}{\sec\theta - \csc\theta}$ is identically equal to $\frac{\sin\theta}{\sin\theta - \cos\theta}$.

   (b) Adding to the definitions of part (a) the definitions that $\tan\theta$ is $\frac{\sin\theta}{\cos\theta}$ and $\cot\theta$ is the reciprocal of $\tan\theta$.

   (i) show that the expression $\frac{\tan\theta + \sec\theta}{\sin\theta\cot\theta}$ can be changed to $\frac{1 + \sin\theta}{\cos^2\theta}$;

   (ii) show that $\frac{1 + \cot\theta}{\csc\theta}$ and $\frac{1 + \tan\theta}{\sec\theta}$ can both be changed to $\sin\theta + \cos\theta$; and

   (iii) show that $\sin\theta\csc\theta$, $\cos\theta\sec\theta$, and $\tan\theta\cot\theta$ are all equal to 1.

5. (a) Formula (1) $\sin^2\theta + \cos^2\theta = 1$ can be used to an advantage in changing the form of many trigonometric expressions. Using this relationship,

   (i) show that $\cos^2\theta - \sin^2\theta$ can be written as $2\cos^2\theta - 1$ or $1 - 2\sin^2\theta$;

   (ii) show that $\tan\theta + \cot\theta$ can be expressed as $\frac{\csc\theta}{\cos\theta}$

   (iii) show that $\frac{1}{1 - \cos\theta} + \frac{1}{1 + \cos\theta}$ can be expressed as $2\csc^2\theta$; and

   (iv) show that both $\cot\theta\csc\theta$ and $\frac{1}{\sec\theta - \cos\theta}$ are both equivalent to $\frac{\cos\theta}{\sin^2\theta}$

   (b) It is simple to prove $\sin\theta\cot\theta = \cos\theta$ and $\cos\theta\tan\theta = \sin\theta$. (Why?) With these relationships, those of Exercise 4(b)(iii), and others developed earlier, prove the following:

   (i) $(1 - \sin^2\theta)\sec^2\theta = 1$

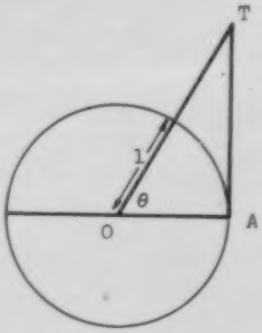
   (ii) $(1 - \cos^2\theta)\csc^2\theta = 1$

   (iii) $\cot^2\theta\,(1 - \cos^2\theta) = \cos^2\theta$

   (iv) $\sec^2\theta\,(1 - \cos^2\theta) = \tan^2\theta$

6. (a) Starting with the relationship $\sin^2\theta + \cos^2\theta = 1$, prove analytically that $1 + \tan^2\theta = \sec^2\theta$.

   (b) By considering ΔOAT and the unit circle to the right, prove geometrically that $1 + \tan^2\theta = \sec^2\theta$.



(c) Starting with the relationship, $\sin^2\theta + \cos^2\theta = 1$, prove analytically that $1 + \cot^2\theta = \csc^2\theta$.

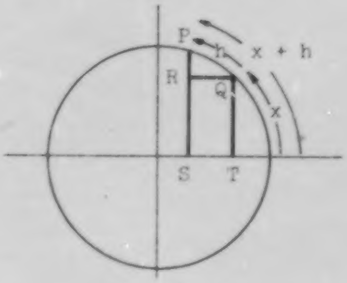
(d) Adding the relationships posed as problems in parts a, b, c to the earlier ones discussed,

(i) show that $\frac{\sec\theta}{\cos\theta} - \frac{\tan\theta}{\cot\theta}$ is identically 1;

(ii) establish that $\sec^2\theta + \csc^2\theta$ is equivalent to $\sec^2\theta\csc^2\theta$ in two ways; and

(iii) show that $\sin^2\theta\,(1 + \cot^2\theta) + \cos^2\theta(1 + \tan^2\theta)$ is always 2.

7. (a) Using the figure to the right prove $\sin(x + h) - \sin x < PQ$.



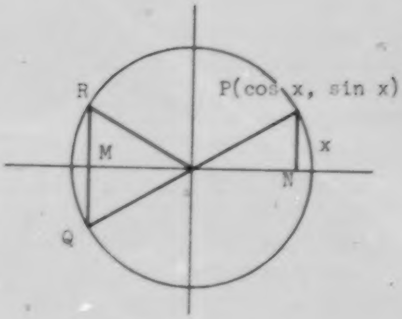
(b) From this result prove that

$|\sin(x + h) - \sin(x)| < |h|$.

(c) Again, using the figure to the right prove that

$|\cos(x + h) - \cos(x)| < |h|$.

8. $P(\cos x, \sin x)$ and $Q(\cos(x + \pi), \sin(x + \pi))$ are indicated on the drawing to the right.



(a) By the use of similar triangles, read off the coordinates of Q; i.e., prove

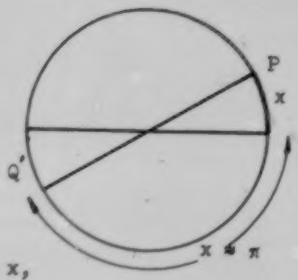
$\cos(x + \pi) = -\cos x$, and
$\sin(x + \pi) = -\sin x$.

(b) Similarly, prove

$\cos(\pi - x) = -\cos x$, and
$\sin(\pi - x) = \sin x$.

9. (a) Using the figure to the right read off the coordinates of Q' to show that

$$\cos(x - \pi) = -\cos x, \text{ and}$$
$$\sin(x - \pi) = -\sin x.$$



(b) Use formula (6) to extend the results of (a) to show that

(i) $\cos(x - \pi) = \cos(\pi - x) = -\cos x$,

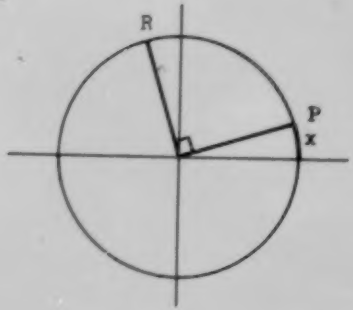
and

(ii) $\sin(x - \pi) = -\sin(\pi - x) = -\sin x$.

10. Read off the coordinates of R to show

$$\cos(x + \tfrac{\pi}{2}) = -\sin x,$$

$$\sin(x + \tfrac{\pi}{2}) = \cos x.$$



11. Using the relationships (6) $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$; and (7) $\sin(\frac{\pi}{2} - x) = \cos x$, $\cos(\frac{\pi}{2} - x) = \sin x$,

(a) prove $\cos(x + \frac{\pi}{2}) = -\sin x$;

(b) prove

(i) $\cos(x + \pi) = -\cos x$,

and

(ii) $\sin(x + \pi) = -\sin x$;

(c) prove

(i) $\cos(x - \pi) = \cos(\pi - x) = -\cos x$

(ii) $\sin(x - \pi) = -\sin(\pi - x) = -\sin x$

12. The inequality $0 \leq 1 - \cos x \leq \frac{x^2}{2}$ was established as formula (3) in this section. By numerical substitution of various values of x, let us now investigate this relationship.

    (a) Using the table which gives the cosine of angles expressed in radian measure, complete the following table.

x (in radians)	$\cos x$	$1 - \cos x$	$\frac{x^2}{2}$
0			
0.1			
0.15			
0.3			
0.5			
0.6			
0.7			
0.8			
0.9			
1.0			
1.2			
1.5			
2			
4			
6			

    (b) From the completed table, conjecture for which values of x this inequality is most useful.

*13. We know that the functions $x \to \cos x$ and $x \to \sin x$ have period 2π. Find the period of the functions $x \to$

    (a) $\sin 2x$
    (b) $\sin \frac{1}{2}x$
    (c) $\cos 4x$
    (d) $\cos \frac{1}{2}x$

*14. Show that the functions sine and cosine have no positive period less than 2π.

3-3. Graphs of the Circular Functions

The sine and cosine functions have been defined in terms of arc length on the unit circle given by $u^2 + v^2 = 1$. As was the case for polynomial functions, the graphs of these functions provide another geometric device for understanding their behavior. At this point for nonpolynomial functions our primary procedure for graphing is the plotting of points. Fortunately we can make use of the results of the previous section to simplify our procedures.

We first plot some points for

(1) $$y = \sin x, \quad 0 \leq x \leq \pi.$$

Table 3-3a lists some values of $\sin x$ which were obtained in the previous section. These points are plotted in Figure 3-3a.

Table 3-3a

Values of $y = \sin x$

x	$y = \sin x$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx .71$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} \approx .87$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2} \approx .87$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} \approx .71$
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	0

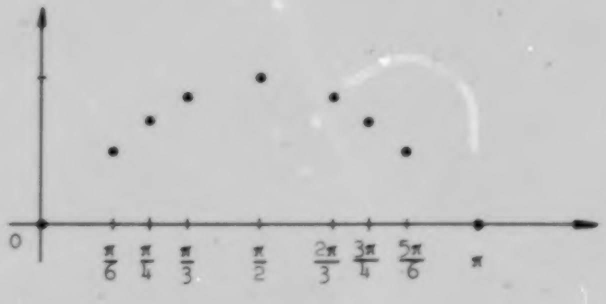


Figure 3-3a. Values of $y = \sin x$, plotted from Table 3-3a.

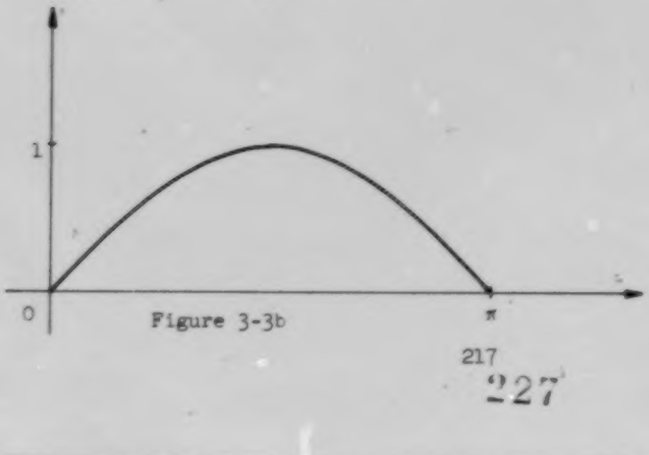


Figure 3-3b

If we connect these points with a smooth curve we obtain the graph shown in Figure 3-3b. A more complete picture can be obtained using more points but this will suffice for our present purposes.

Now we can make use of the properties obtained in the previous section to extend our graph beyond the interval $0 \leq x \leq \pi$. The identity

$$(2) \qquad \sin(-x) = -\sin x$$

tells us that the graph is symmetric with respect to the origin; that is, the graph contains $(-x,-y)$ if it contains (x,y). (Such a function is also called an odd function. Later we shall show how to approximate the sine function by a polynomial function with only odd degree terms.) Equation (2) enables us to obtain Figure 3-3c from Figure 3-3b.

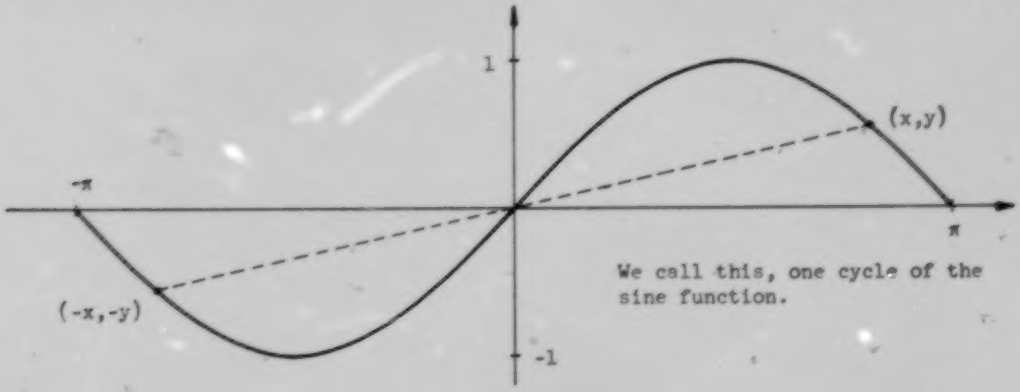


Figure 3-3c. $y = \sin x, -\pi \leq x \leq \pi$

Next we use the identity

$$(3) \qquad \sin(x + 2n\pi) = \sin x,$$

which holds for all integers n and all real numbers x, to obtain the graph shown in Figure 3-3d. The identity states algebraically that the graph of the sine function coincides with itself under a translation of $2n\pi$ units (to the right if n is a negative integer and to the left if n is a positive integer).

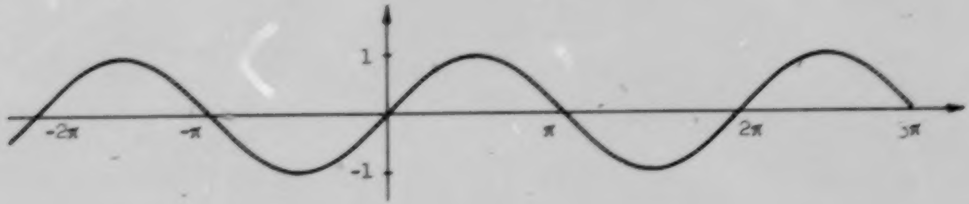


Figure 3-3d. $y = \sin x$

The graph of the cosine function can be obtained in a similar manner, for we know that

$$\cos x = \sin\left(x + \frac{\pi}{2}\right). \quad (4)$$

Thus we can picture a shift of the graph of the sine function $\frac{\pi}{2}$ units to the left to obtain the graph of the cosine function.

In Figure 3-3e we indicate this relationship by superimposing on the same axes the graphs of the sine and cosine functions.

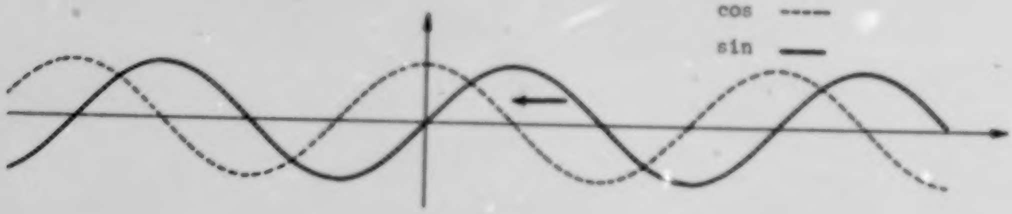


Figure 3-3e

Translation and Stretching

We have observed that the cosine function can be obtained from the sine function by translation. This process generalizes. The graph of

$$y = \sin(x + C)$$

where C is a constant is easily obtained by suitably translating the graph of $y = \sin x$ on the x-axis. We can think of shifting the graph (in Figure 3-3d) $|C|$ units to the right or left according as C is negative or positive. For example, in Figure 3-3f we show the graphs of $y = \sin(x + 2)$ and $y = \sin(x - \frac{\pi}{4})$ under the graph of $y = \sin x$, to show how each can be obtained from the graph of $y = \sin x$ by an appropriate translation.

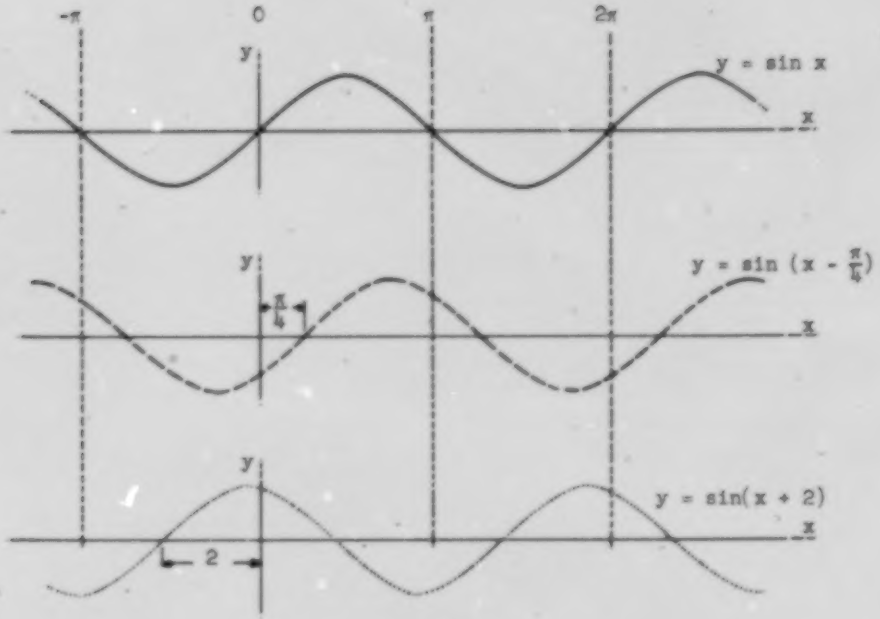


Figure 3-3f

In Figure 3-3g we picture the graphs of $y = \sin x$, $y = 2 \sin x$, and $y = \sin 2x$.

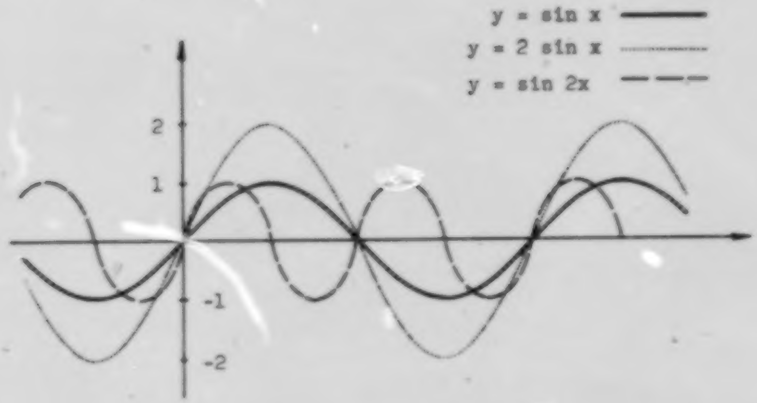


Figure 3-3g

We can describe the graph of $y = 2 \sin x$ as being obtained from the graph of $y = \sin x$ by "stretching" each ordinate by a factor of 2, and similarly, the graph of $y = \sin 2x$ being obtained by "shrinking" each abscissa.

The graph of

$$y = A \sin (Bx + C),$$

called the general sinusoidal curve, can be obtained by combining translation and scale change. For example, to graph

$$y = 3 \sin \left(2x + \frac{\pi}{2}\right) \qquad (5)$$

we observe that

$$\sin \left(2x + \frac{\pi}{2}\right) = \sin \left(2\left(x + \frac{\pi}{4}\right)\right)$$

so that the graph of (5) can be obtained from $y = 3 \sin 2x$ by shifting the graph $\frac{\pi}{4}$ units to the left. The graph of $y = 3 \sin 2x$ can be obtained from that of $y = \sin x$ by "stretching" each ordinate by a factor of 3 and "shrinking" each abscissa by a factor of 2. (See Figure 3-3h.)

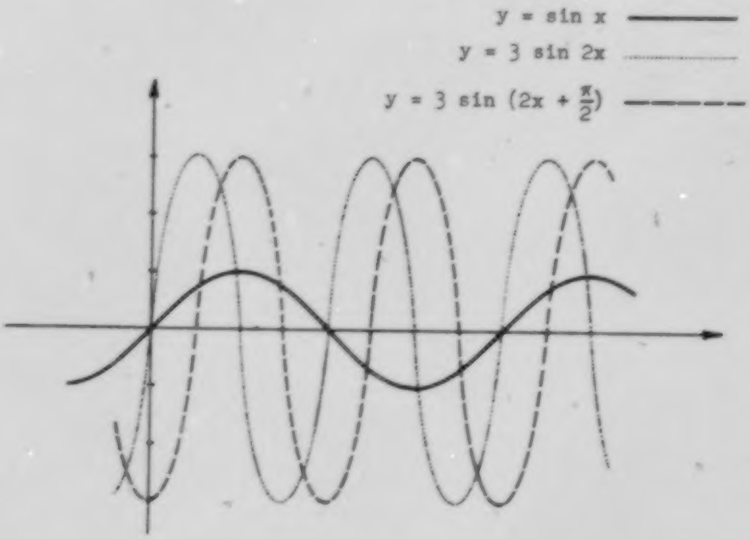


Figure 3-3h

Exercises 3-3

For each of the following, sketch the graphs of the three equations on one set of axes over the interval(s) indicated.

1. (a) $y = 2 \cos x$ $\quad 0 \leq x \leq 2\pi$

   (b) $y = 3 \cos x$ $\quad 0 \leq x \leq 2\pi$

   (c) $y = \frac{1}{2} \cos x$ $\quad 0 \leq x \leq 2\pi$

2. (a) $y = \cos 2x$ $\quad 0 \leq x \leq 2\pi$

   (b) $y = \cos 3x$ $\quad 0 \leq x \leq 2\pi$

   (c) $y = \cos \frac{1}{2} x$ $\quad 0 \leq x \leq 4\pi$

3. (a) $y = \cos (x + \frac{\pi}{2})$ $\quad -\frac{\pi}{2} \leq x \leq 3\pi$

   (b) $y = \cos (x - \frac{\pi}{2})$ $\quad \frac{\pi}{2} \leq x \leq 3\pi$

   (c) $y = \cos (x + \pi)$ $\quad -\pi \leq x \leq 3\pi$

4. (a) $y = -\cos x$ $\quad 0 \leq x \leq 2\pi$

   (b) $y = -2 \cos x$ $\quad 0 \leq x \leq 2\pi$

   (c) $y = -\cos 2x$ $\quad 0 \leq x \leq 2\pi$

5. (a) $y = -\sin 2x$ $\quad 0 \leq x \leq 2\pi$

   (b) $y = -2 \sin 4x$ $\quad 0 \leq x \leq 2\pi$

   (c) $y = -\frac{5}{2} \sin \frac{8}{3} x$ $\quad 0 \leq x \leq 2\pi$

6. (a) $y = -\cos (x - \frac{\pi}{2})$ $\quad 0 \leq x \leq 4\pi$

   (b) $y = \sin (x + \pi)$ $\quad 0 \leq x \leq 4\pi$

   (c) $y = \cos (x + \frac{\pi}{2})$ $\quad 0 \leq x \leq 4\pi$

7. (a) $y - 1 = \cos x$ $\quad 0 \leq x \leq 4\pi$

   (b) $y + 2 = \sin \frac{x}{2}$ $\quad 0 \leq x \leq 4\pi$

   (c) $y + \frac{1}{2} = \frac{1}{2} \sin 2x$ $\quad 0 \leq x \leq 4\pi$

8. (a) $y = |\sin x|$ $\quad 0 \le x \le 2\pi$

   (b) $y = \frac{1}{2}|\sin 2x|$ $\quad 0 \le x \le 2\pi$

   (c) $y = \frac{3}{2}|\sin \frac{4}{5} x|$ $\quad 0 \le x \le 2\pi$

9. (a) $y = -|\cos x|$ $\quad 0 \le x \le 2\pi$

   (b) $y = |\sin (x - \frac{\pi}{2})|$ $\quad 0 \le x \le 2\pi$

   (c) $y = |\sin (x - \frac{\pi}{2})| - |\cos x|$ $\quad 0 \le x \le 2\pi$

10. (a) $y = \sin^2 x$ $\quad 0 \le x \le 2\pi$

    (b) $y = \cos^2 x$ $\quad 0 \le x \le 2\pi$

    (c) $y = \sin^2 x + \cos^2 x$ $\quad 0 \le x \le 2\pi$

3-4. Uniform Circular Motion

Let us now consider the motion of a point P around a circle of radius r in the uv-plane, and suppose that P moves at the constant speed of s units per second. We let $P_0(r,0)$ represent the initial position of P. After one second P will be an arc-distance s units away from P_0; after 2 seconds P will be an arc-distance $2s$ units away from P_0; and similarly after t seconds P will be an arc-distance ts units from its starting point $(r,0)$. In Figure 3-4a we show a point $P(u,v)$, which is st units from P_0 (measured clockwise if $st > 0$) around the circle given by $u^2 + v^2 = r^2$.

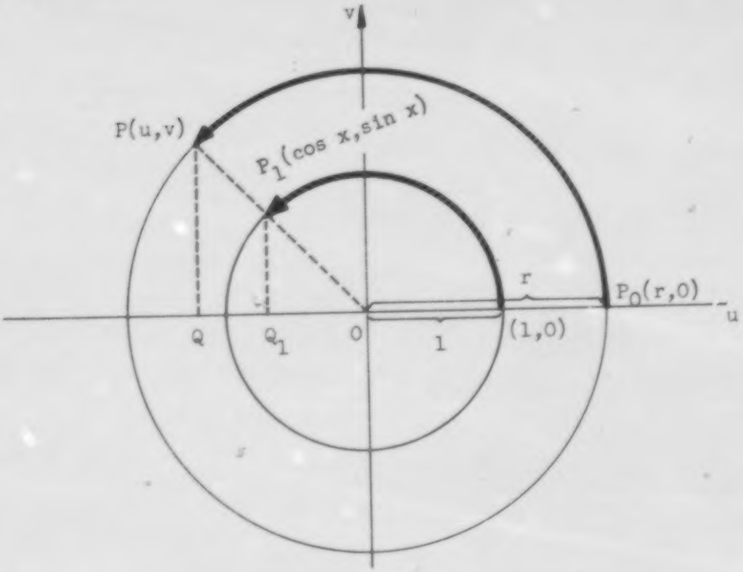


Figure 3-4a

We wish to describe the coordinates (u,v) of P in terms of values of the sine and cosine functions. Since we defined the functions $x \to \sin x$ and $x \to \cos x$ in terms of a unit circle, we also draw the circle given by $u^2 + v^2 = 1$ in Figure 3-4a. (While we illustrate the case where $r > 1$, our reasoning will also hold for the case where $r < 1$.) The line $\overline{OP}$ crosses the unit circle at the point $P_1(\cos x, \sin x)$.

We can express these coordinates in terms of t instead of x. What happens to st and x if t is doubled, tripled, halved, or multiplied by some constant factor k? What is x when $st = 2\pi r$? We know that st is directly proportional to t. It follows that x is directly proportional to t; that is, if k is a constant, $x = kt$. When P has completely traversed the circle given by $u^2 + v^2 = r^2$, then $st = 2\pi r$; it follows that when $st = 2\pi r$ we have $x = 2\pi$ (since the unit circle has circumference 2π units). Thus, we have

$$2\pi = k\left(\frac{2\pi r}{s}\right),$$

from which it follows that $k = \frac{s}{r}$. Alternatively we could reason that

$$\frac{st}{2\pi r} = \frac{x}{2\pi} .$$

Since $x = kt$, we get

$$\frac{st}{2\pi r} = \frac{kt}{2\pi} ;$$

whence we arrive at the same result: $k = \frac{s}{r}$.

To summarize we can say that the coordinates (u,v) at any time t seconds are given by

$$u = r \cos \left(\frac{s}{r}\right)t$$

and

$$v = r \sin \left(\frac{s}{r}\right)t.$$

The constant of proportionality $k = \frac{s}{r}$ is commonly denoted by ω and is called the angular velocity of P. It is called angular velocity because the measure of any central angle P_0OP (Figure 3-4a) may be written as $\omega = \frac{s}{r}$. In t seconds OP rotates through an angle measure of ωt as P moves an arc-distance of st units. If we let $\omega = \frac{s}{r}$, we can write

$$\begin{aligned} u &= r \cos \omega t \\ v &= r \sin \omega t. \end{aligned} \tag{1}$$

When $\omega t = 2\pi$, P will again be in the position P_0. This motion of the point from P_0 back into P_0 again is called a cycle. The time interval during which a cycle occurs is called the period; in this case, the period is $\frac{2\pi}{\omega}$. The number of cycles which occur during a fixed unit of time is called the frequency. We give a commonplace example of frequency when we refer to the alternating current in our homes as "60-cycle", an abbreviation for "60 cycles per second."

Example 3-4a. Consider the motion of a point P around a circle of radius 2 in the uv-plane. Suppose that P moves at the constant speed of 3 units per second. Since $\omega = \frac{s}{r} = \frac{3}{2}$, the angular velocity is $\frac{3}{2}$ units per second; the coordinates of $P(u,v)$ are given by

$$u = 2 \cos \left(\frac{3}{2}\right)t \quad \text{and} \quad v = 2 \sin \left(\frac{3}{2}\right)t;$$

the period is $\frac{2\pi}{\omega} = \frac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}$; and the frequency is $\frac{3}{4\pi}$.

To visualize the behavior of the point P in a different way, consider the motion of the point Q which is the projection of P on the v-axis. As P moves around the unit circle, Q moves up and down along a fixed diameter of the circle, and a pencil attached to Q will trace this diameter repeatedly -- assuming that the paper is fixed in position. If, however, the strip of paper is drawn from right to left at a constant speed, then the pencil will trace a curve, something like Figure 3-4b.

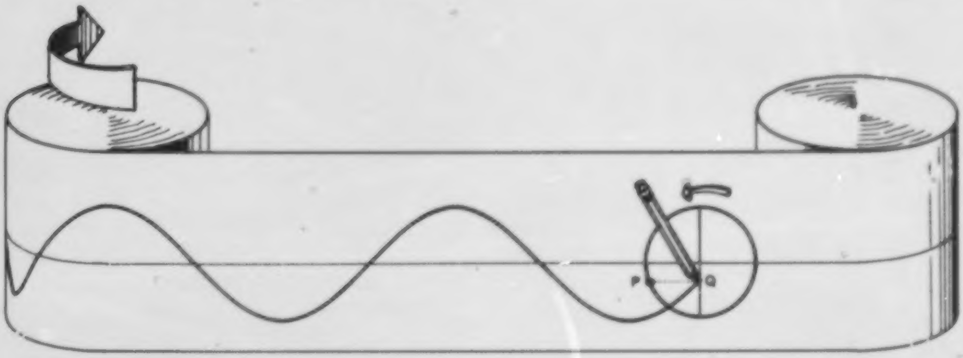


Figure 3-4b. Wave Motion

An examination of this figure will show why motion of this type is called wave motion. We note that the displacement y of Q from its central position is functionally related to the time t, that is, there is a function f such that $y = f(t)$. By suitably locating the origin of the ty-plane, we may have either $y = \cos \omega t$ or $y = \sin \omega t$; thus either of these equations may be looked upon as describing a pure wave or, as it is sometimes called, a simple harmonic motion. The surface of a body of water displays a wave motion when it is disturbed. Another familiar example is furnished by the electromagnetic waves used in radio, television, and radar, and modern physics has even detected wave-like behavior of the electrons of the atom.

One of the most interesting applications of the circular functions is to the theory of sound (acoustics). A sound wave is produced by a rapid alternation of pressure in some medium. A pure musical tone is produced by any pressure wave which can be described by a circular function of time, say:

$$p = A \sin \omega t \tag{2}$$

where p is the pressure at time t and the constants A and ω are positive. The equation (2) for the acoustical pressure, p, is exactly in the form of one of the equations of (1) even though no circular motion is involved; all that occurs is a fluctuation of the pressure at a given point of space.* Here the numbers A and ω have direct musical significance. The positive number A is called the amplitude of the wave; it is the peak pressure and its square is a measure of the loudness. The number ω is proportional to the frequency and is a measure of pitch; the larger ω the more shrill the tone.

The effectiveness of the application of circular functions to the theory of sound stems from the principle of superposition. If two instruments individually produce acoustical pressures p_1 and p_2 then together they produce the pressure $p_1 + p_2$. If p_1 and p_2 have a common period then the sum $p_1 + p_2$ has the same period. This is the root of the principle of harmony; if two instruments are tuned to the same note, they will produce no strange new note when played together.

Let us suppose, for example, that two pure tones are produced with individual pressure waves of the same frequency, say

$$u = A \cos \omega t \tag{3}$$

$$v = B \sin \omega t \tag{4}$$

where A, B and ω are positive. According to the principle of superposition, the net pressure is

$$p = A \cos \omega t + B \sin \omega t.$$

What does the graph of this equation look like? We shall answer this question by reducing the problem to two simpler problems, that is, of graphing (3) and (4) above. For each t, the value of p is obtained from the individual graphs, since

---

*The acoustical pressure is defined as the difference between the gas pressure in the wave and the pressure of the gas if it is left undisturbed.

$$p = u + v.$$

To illustrate these ideas with specific numerical values in place of A, B and ω; let

$$A = 3, \quad B = 4, \quad \omega = \pi.$$

Then we wish to graph

$$p = 3 \cos \pi t + 4 \sin \pi t. \tag{5}$$

Equations (3) and (4) become

$$u = 3 \cos \pi t, \tag{6}$$

$$v = 4 \sin \pi t. \tag{7}$$

By drawing the graphs of (6) (Figure 3-4c) and (7) (Figure 3-4d) on the same set of axes, and by adding the corresponding ordinates of these graphs at each value of t, we obtain the graph of (5) shown in Figure 3-4e. You will notice that certain points on the graph of p are labeled with their coordinates. These are points which are either easy to find, or which have some special interest.

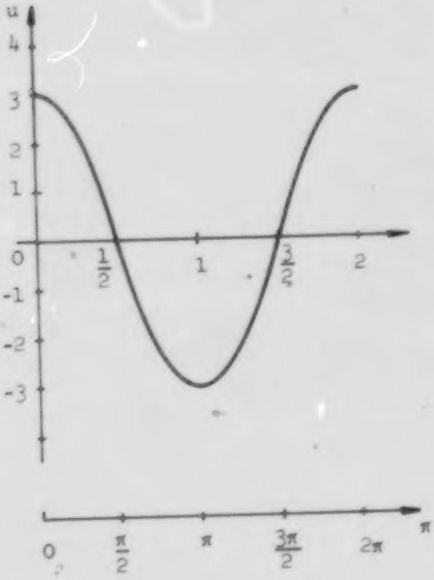


Figure 3-4c. Graph of $u = 3 \cos \pi t$.

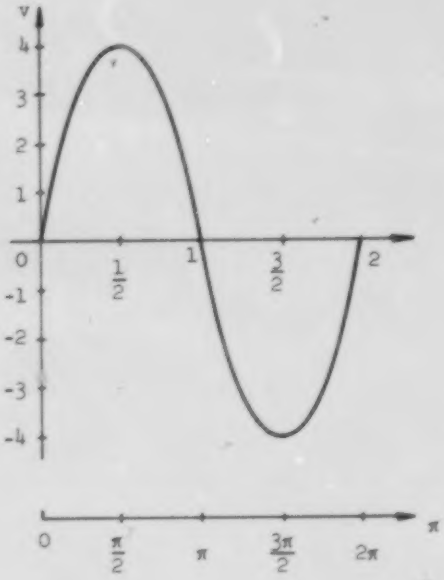


Figure 3-4d. Graph of $v = 4 \sin \pi t$.

The points (0,3), (0.5,4), (1,-3), (1.5,-4) and (2,3) are easy to find since they are the points where either $u = 0$ or $v = 0$. The points (0.29, 5) and (1.29, -5) are important because they represent the first maximum and minimum points on the graph of p, while (0.79, 0) and (1.79, 0) are the first zeros of p. To find the maximum and minimum points and zeros of p involves the use of tables and hence we shall put off a discussion of this matter until Section 3-6, although a careful graphing should produce fairly good approximations to them.

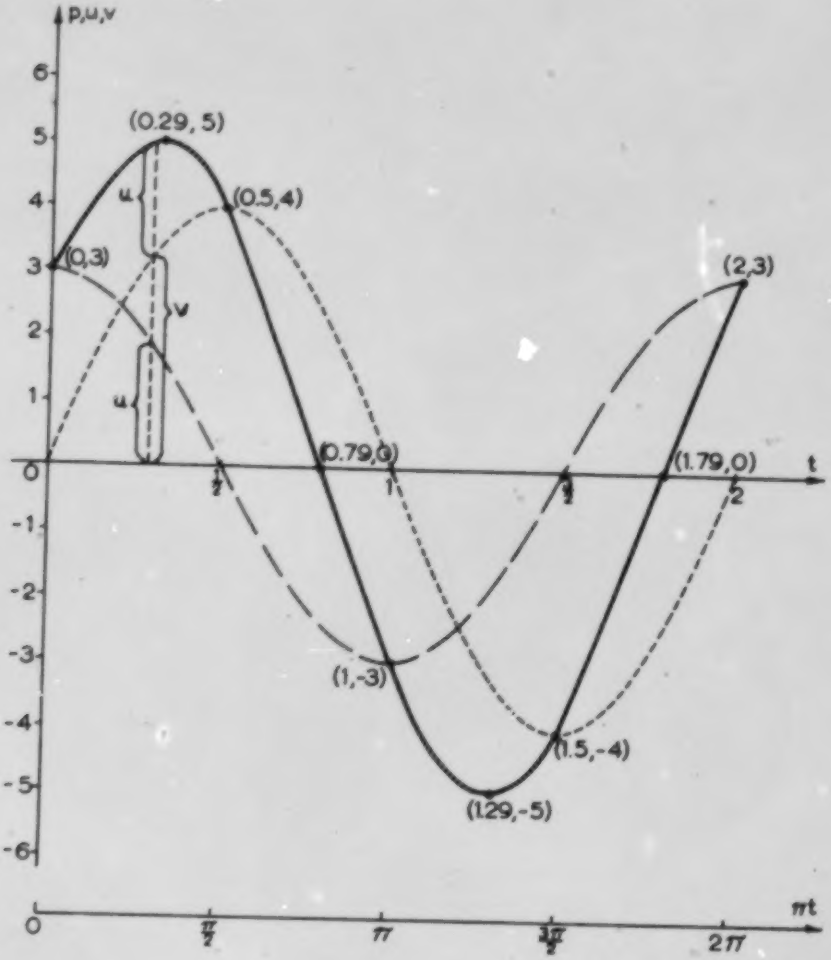


Figure 3-4e. The sum of two pure waves of equal period.

Dashed curve: $u = 3 \cos \pi t$. Dotted curve: $v = 4 \sin \pi t$. Full curve: $p = 3 \cos \pi t + 4 \sin \pi t$; $0 \leq t \leq 2$. (The scales are not the same on the two axes; this distortion is introduced in order to show the details more clearly.)

Exercises 3-4

1. Sketch graphs of each of the following curves over one complete cycle; and state what the period is, and what the range is, if you can.

   (a) $y = 2 \sin 3t$

   (b) $y = -3 \sin 2t$

   (c) $y = 4 \cos (\frac{x}{2})$

   (d) $y = 3 \cos (-x)$

   (e) $y = 2 \sin x - \cos x$

2. (a) Find the length of the arc traversed when $\omega = \frac{2\pi}{3}$, $r = 3$, if

   (i) $t = 4$ (iii) $t = 6$

   (ii) $t = 2$ (iv) $t = t_0$

   (b) For a given ω in a circle, how is the arc length affected if the radius is doubled? tripled?

   (c) Find the length of the arc traversed when $\omega = \frac{3\pi}{4}$, $t = 3$ if

   (i) $r = 5$ (iii) $r = 10$

   (ii) $r = \frac{5}{2}$ (iv) $r = R$

   (d) For a given ω and a given time, how is the arc length affected if the radius is halved? doubled?

   (e) Find the length of the arc traversed under $r = 10$, $t = 4\frac{1}{2}$ if

   (i) $\omega = \frac{\pi}{6}$ (iii) $\frac{2\pi}{3}$

   (ii) $\frac{\pi}{3}$ (iv) φ

   (f) If the time is given and the circle fixed how is the length of the arc affected if ω is doubled? quadrupled?

3. For the following, sketch and identify the

   (i) period

   (ii) location of maximum point(s) and

   (iii) minimum points in this interval $0 \leq x \leq 2\pi$.

   (a) $y = -\frac{1}{2} \sin 2x$ (c) $y = 2\left|\cos \frac{3x}{2}\right|$

   (b) $y = 2 \cos \frac{3x}{2}$

4. (a) On one set of axes, using the same scale over the interval $0 \leq x \leq 2\pi$. Sketch the graphs of

   (i) $y = \sin x$

   (ii) $y = \cos x$

(b) (i) Using the sketches and the scale in part (a), sketch on the same graph $y = \sin x + \cos x$.

   (ii) From the graph of $y = \sin x + \cos x$, conjecture the period, and the maximum and minimum point(s).

(c) (i) Sketch $y = \sqrt{2}\cos(x - \frac{\pi}{4})$ using same scale as (b)(i).

   (ii) Sketch $y = \sqrt{2}\sin(x + \frac{\pi}{4})$ using same scale as (b)(i).

(d) Compare the graphs of (b) and (c).
Have you any conjectures?

3-5. The Addition Formulas

In Section 3-4 we added the corresponding ordinates of the graphs of $t \rightarrow 3 \cos \pi t$ and $t \rightarrow 4 \sin \pi t$ at each value of t to obtain the graph of

$$f : t \rightarrow p = 3 \cos \pi t + 4 \sin \pi t$$

over the interval $0 \leq t \leq 2$. We could have obtained the graph of f more easily if we had been able to express f in the form

$$f : t \rightarrow p = A \sin (\pi t + \alpha).$$

In this section we shall derive formulas which will enable us to show that, for all real values of t,

$$3 \cos \pi t + 4 \sin \pi t = A \sin (\pi t + \alpha),$$

where $A = 5$, $\cos \alpha = \frac{4}{5}$, and $\sin \alpha = \frac{3}{5}$.

The formulas that we shall derive will also help us to discuss tangent lines to the graphs of circular functions and areas beneath them.

We begin by deriving the basic formula

(1) $$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

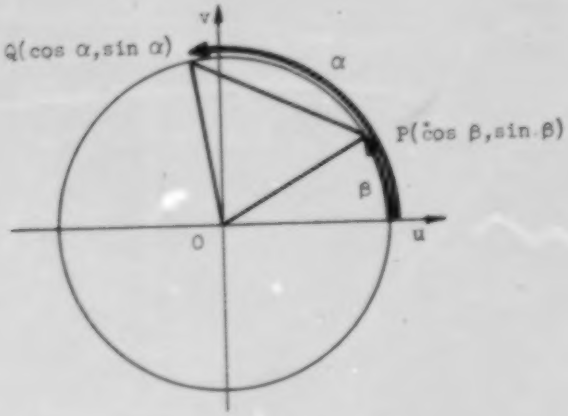


Figure 3-5a.

You may have derived this formula in an earlier course. To begin our derivation we refer to Figure 3-5a. (We illustrate the case for which $0 < \beta < \alpha$.) The distance from P to Q is

$$(2) \qquad PQ = \sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2}\ .$$

We now use the principle that arclength on a circle depends only upon the unit of measure and not on the choice of axes. If we choose the u' and v' axes (Figure 3-5b), we see that P now has coordinates $(1,0)$ and Q has the coordinates $(\cos(\alpha - \beta), \sin(\alpha - \beta))$.

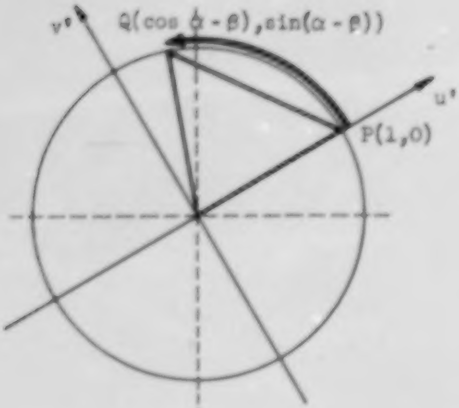


Figure 3-5b

In Figure 3-5b the distance from P to Q is

$$(3) \qquad PQ = \sqrt{(1 - \cos(\alpha - \beta))^2 + (0 - \sin(\alpha - \beta))^2}$$

We equate this with (2) and square both sides to obtain

$$(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 = [1 - \cos(\alpha - \beta)]^2 + [0 - \sin(\alpha - \beta)]^2.$$

Expanding and regrouping, we get on the left

$$(\cos^2\beta + \sin^2\beta) + (\cos^2\alpha + \sin^2\alpha) - 2(\cos\beta\cos\alpha + \sin\beta\sin\alpha)$$

and on the right

$$1 + [\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)] - 2\cos(\alpha - \beta).$$

Since, for all real x, $\sin^2 x + \cos^2 x = 1$, we have

$$1 + 1 - 2(\cos\alpha\cos\beta + \sin\beta\sin\alpha) = 1 + 1 - 2\cos(\alpha - \beta).$$

Therefore, we conclude that (1) holds; i.e.,

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\beta\sin\alpha.$$

While we could use a similar argument to derive the formula for $\cos(\alpha + \beta)$, we elect to use (1). Replacing β by $-\beta$ in (1) we have

$$\begin{aligned}\cos(\alpha + \beta) &= \cos(\alpha - (-\beta)) \\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta).\end{aligned}$$

Since $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$ we have

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{4}$$

Earlier we showed that, for all real x,

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) \quad \text{and} \quad \cos x = \sin\left(\frac{\pi}{2} - x\right). \tag{5}$$

We can use (1) and (5) to obtain

$$\begin{aligned}\sin(\alpha + \beta) &= \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right] \\ &= \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right] \\ &= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta\end{aligned}$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \tag{6}$$

Replacing β by $-\beta$ in (6) we get

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{7}$$

The following examples show some of the many formulas which can be derived from the foregoing addition (sum and difference) formulas.

Example 3-5a. Show that for all real x

$$\cos^2 x = \frac{1 + \cos 2x}{2}. \tag{8}$$

We use (4) with $\alpha = \beta = x$ to obtain

$$\cos 2x = \cos x\cos x - \sin x\sin x = \cos^2 x - \sin^2 x.$$

Since $\cos^2 x + \sin^2 x = 1$, we can rewrite this as

$$\cos 2x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1.$$

Solving for $\cos^2 x$ we get (8).

Example 3-5b. Show that for all real x

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = \sin x + \cos x \tag{9}$$

We use (6) with $\alpha = x$, $\beta = \frac{\pi}{4}$ to obtain

$$\sin\left(x + \frac{\pi}{4}\right) = \cos x \sin\frac{\pi}{4} + \sin x \cos\frac{\pi}{4}.$$

Since

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

we get

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\cos x + \sin x).$$

Therefore, $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$. A slight generalization of this process will be used in Section 3-6 to rewrite (from Section 3-4)

$$3\cos\pi t + 4\sin\pi t$$

as $5\sin(\pi t + \alpha)$, where $\cos\alpha = \frac{4}{5}$, $\sin\alpha = \frac{3}{5}$.

Example 3-5c. Show that for all real numbers a, b, and x

$$\sin ax \cos bx = \frac{1}{2}[\sin((a + b)x) + \sin((a - b)x)]. \tag{10}$$

We let $\alpha = ax$ and $\beta = bx$ to obtain

(11) $\sin((a + b)x) = \sin(ax + bx) = \cos ax \sin bx + \sin ax \cos bx.$

Formula (7) gives:

(12) $\sin((a - b)x) = \sin(ax - bx) = \sin ax \cos bx - \cos ax \sin bx.$

Adding (11) and (12) we get

$$\sin((a + b)x) + \cos((a + b)x) = 2\sin ax \cos bx.$$

Dividing by 2 we obtain (10).

Exercises 3-5

1. Show that for all real x

   (a) $\sin 2x = 2 \sin x \cos x$

   (b) $$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\end{aligned}$$

   (c) $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

2. Sketch $(0 \leq x \leq 2\pi)$ and show that

   (a) $$\begin{aligned}\cos x + \sin x = \sqrt{2}\sin\left(x + \tfrac{\pi}{4}\right) &= \sqrt{2}\cos\left(x - \tfrac{\pi}{4}\right)\\ &= \sqrt{2}\cos\left(x + \tfrac{7\pi}{4}\right)\end{aligned}$$

   (b) $$\begin{aligned}\cos x - \sin x = \sqrt{2}\cos\left(x + \tfrac{\pi}{4}\right) &= -\sqrt{2}\sin\left(x - \tfrac{\pi}{4}\right)\\ &= \sqrt{2}\sin\left(x + \tfrac{3\pi}{4}\right)\end{aligned}$$

3. Using formulas (1), (4), (6), (7) and Exercise 1 show that

   (a) $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

   (b) $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

   (c) $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

   (d) $\tan\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1 - \cos\alpha}{1 + \cos\alpha}} = \dfrac{1 - \cos\alpha}{\sin\alpha} = \dfrac{\sin\alpha}{1 + \cos\alpha}$

4. Use the law of cosines to derive formula (3).

5. Show that for all numbers a, b, and x

   (a) $\sin ax \sin bx = \frac{1}{2}[\cos(a - b)x - \cos(a + b)x]$

   (b) $\cos ax \cos bx = \frac{1}{2}[\cos(a - b)x + \cos(a + b)x]$

6. Using any of the formulas developed in this chapter, find:

   (a) $\sin \frac{\pi}{12}$ (Hint: $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$)

   (b) $\cos \frac{5\pi}{12}$

   (c) $\tan \frac{7\pi}{12}$

   (d) $\cos \frac{11\pi}{12}$

7. Using any of the formulas developed in this chapter, show that for all values where the functions are defined the following are identities:

   (a) $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$

   (b) $\cos^2 \frac{1}{2}\theta = \frac{\tan \theta + \sin \theta}{2 \tan \theta}$

   (c) $1 + \sin \alpha = (\sin \frac{1}{2}\alpha + \cos \frac{1}{2}\alpha)^2$

   (d) $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$

   (e) $\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$

   (f) $\frac{1 + \cos \theta}{\sin \theta} + \frac{\sin \theta}{1 + \cos \theta} = \frac{2}{\sin \theta}$

8. You derived the formula:

$$\cos 2x = 2 \cos^2 x - 1.$$

   (a) Solve this for $\cos^2 x$ thus expressing $\cos^2 x$ as a linear function of $\cos 2x$.

   (b) Consider $\cos^4 x$ as $(\cos^2 x)^2$ and by the same methods as used in (a) show that

$$\cos^4 x = \frac{1}{8}(3 + 4 \cos 2x + \cos 4x).$$

9. Using the formula $\cos 2x = 1 - 2 \sin^2 x$, derive the formula for $\sin^4 x$:

$$\sin^4 x = \frac{1}{8}(3 - 4 \cos 2x + \cos 4x).$$

10. Show that the following are identities: that is, they are true for all values for which the functions are defined.

    (a) $\sin 2\theta \cos \theta - \cos 2\theta \sin \theta = \sin \theta$

    (b) $\sin(x - y) \cos z + \sin(y - z) \cos x = \sin(x - z) \cos y$

    (c) $\sin 3x \sin 2x = \frac{1}{2}(\cos x - \cos 5x)$

    (d) $\cos \theta - \sin \theta \tan 2\theta = \frac{\cos 3\theta}{\cos 2\theta}$

    (e) $\sin^3 \theta = \frac{1}{4}(3 \sin \theta - \sin 3\theta)$

    (f) $\sin x + \sin 2x + \sin 3x = \sin 2x (2 \cos x + 1)$

    (g) $\left(\frac{1 + \tan x}{1 - \tan x}\right)^2 = \frac{1 + \sin 2x}{1 - \sin 2x}$

3-6. Pure Waves

We promised that formulas derived in the last section would enable us to write

(1) $$p = 3 \cos \pi t + 4 \sin \pi t$$

in the form

(2) $$p = A \sin (\omega t + \alpha).$$

We apply the formula (6 of Section 3-5) for the sine of the sum of two numbers to (2) to obtain

$$\begin{aligned} A \sin (\omega t + \alpha) &= A \sin \omega t \cos \alpha + A \cos \omega t \sin \alpha \\ &= A \sin \alpha \cos \omega t + A \cos \alpha \sin \omega t. \end{aligned}$$

Now if this is to be the same as (1) we must choose $\omega = \pi$,

(3) $$\begin{aligned} A \sin \alpha &= 3 \\ A \cos \alpha &= 4. \end{aligned}$$

To find A we take the sum of the squares in (3) to obtain

$$\begin{aligned} A^2 \sin^2 \alpha + A^2 \cos^2 \alpha &= 3^2 + 4^2, \\ A^2(\sin^2 \alpha + \cos^2 \alpha) &= 25, \\ A^2 &= 25. \end{aligned}$$

Thus we can choose $A = 5$ and then choose α so that

(4) $$\sin \alpha = \frac{3}{5} \quad \text{and} \quad \cos \alpha = \frac{4}{5} .$$

From tables we get $\alpha \approx .643$.

We have shown that, for all real values of t, we can write

(5) $$3 \cos \pi t + 4 \sin \pi t = 5 \sin(\pi t + \alpha),$$

where $\alpha \approx .643$.

We can use the same procedure which we have followed for our particular numerical example to express any equation of the type

(6) $$y = B \cos \omega t + C \sin \omega t$$

in the form

(7) $$y = A \sin(\omega t + \alpha).$$

By using the sum formula for $\sin(\omega t + \alpha)$ we obtain the two equations

$$A \sin \alpha = B, \quad A \cos \alpha = C,$$

which can be solved by putting

(8) $$A = \sqrt{B^2 + C^2}$$

and choosing α so that

(9) $$\sin \alpha = \frac{B}{A}, \quad \cos \alpha = \frac{C}{A}.$$

We can choose exactly one number α such that $0 \leq \alpha < 2\pi$. We know from (8) that

$$(\frac{C}{A})^2 + (\frac{B}{A})^2 = 1$$

so there is a unique point P with coordinates $(\frac{C}{A}, \frac{B}{A})$ on the circle given by $u^2 + v^2 = 1$. There is then a unique α on the interval $0 \leq \alpha < 2\pi$ so that P is α units around the unit circle from (1,0); that is, so that (9) holds.

Consider the function

(10) $$f : x \rightarrow y = A \sin (\omega t + \alpha),$$

where $A > 0$ and $0 \leq \alpha < 2\pi$. The graph of f is called a pure wave (or sine curve). We call A the amplitude; ω, the phase; and $\frac{2\pi}{\omega}$, the period of the wave. The amplitude A is the maximum value of f and -A is the minimum value of f. The period is the distance between successive maxima (or minima) of f. We can rewrite the equation of (10) as

(11) $$y = A \sin (\omega(t + \frac{\alpha}{\omega})).$$

From this we see that the graph of (10) can be obtained from the graph of

(12) $$y = A \sin \omega t$$

by shifting the y-axis $\frac{\alpha}{\omega}$ units to the right. This information is illustrated in Figure 3-6a for the graph of $p = 5 \sin (\pi t + \alpha)$, where $\alpha \approx .643$.

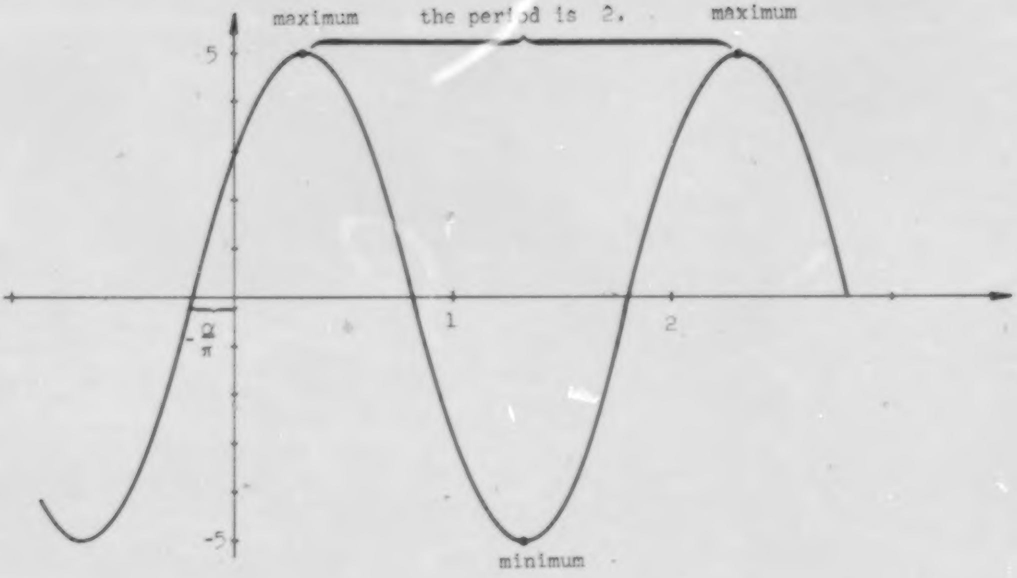


Figure 3-6a

$$y = 5 \sin (\pi t + \alpha),\ \alpha \approx .643$$

We have seen that in general there is a better way to sketch the graph of $y = B \cos \omega t + C \sin \omega t$ than to add the ordinates of the graphs of $y = B \cos \omega t$ and $y = C \sin \omega t$. To discuss a function $t \rightarrow y$ defined by

$$y = B \cos \omega t + C \sin \omega t$$

expediently and to be able to graph it quickly, we can write

$$y = A \sin (\omega t + \alpha).$$

If we write the function in this form we can tell by inspection the period $(\frac{2\pi}{\omega})$, the amplitude $(|A|)$, the maximum and minimum values $(\pm|A|)$. Since

$$A \sin (\omega t + \alpha) = A \sin \omega(t + \frac{\alpha}{\omega}),$$

we can obtain the graph by shifting the graph of

$$y = A \sin \omega t$$

$|\frac{\alpha}{\omega}|$ units to the left or right according as $\frac{\alpha}{\omega}$ is positive or negative.

It would be just as convenient to express $y = B \cos \omega t + C \sin \omega t$ in the form $y = A \cos (\omega t + \beta)$. We leave this for the exercises.

Example 3-6a. We wish to discuss and sketch the graph of the function given by the equation

$$y = 2 \sin \frac{3}{2}t - 3 \cos \frac{3}{2}t. \tag{13}$$

We want to write (13) in the form

$$y = A \sin (\tfrac{3}{2}t + \alpha).$$

Our addition formula enables us to write

$$A \sin (\tfrac{3}{2}t + \alpha) = A \sin \tfrac{3}{2}t \cos \alpha + A \cos \tfrac{3}{2}t \sin \alpha.$$

For all real values of t we require that

$$A \sin \tfrac{3}{2}t \cos \alpha + A \cos \tfrac{3}{2}t \sin \alpha = 2 \sin \tfrac{3}{2}t - 3 \cos \tfrac{3}{2}t.$$

Therefore, we must have

$$A \cos \alpha = 2 \text{ and } A \sin \alpha = -3.$$

Following our earlier procedure we write

$$A^2(\cos^2 \alpha + \sin^2 \alpha) = (2)^2 + (-3)^2,$$

whence we get $A = \sqrt{13}$. Referring to the unit circle of Figure 3-6b, we see that the point $\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\right)$ lies in the fourth quadrant.

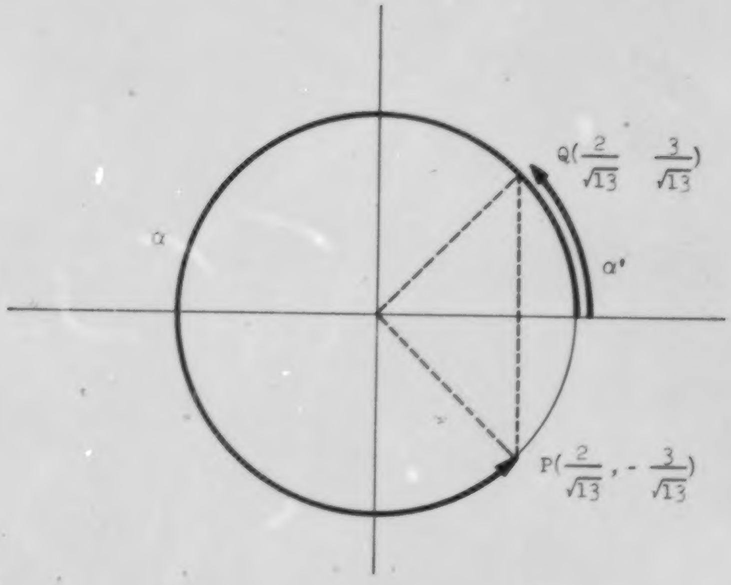

$Q(\frac{2}{\sqrt{13}}\ \ \frac{3}{\sqrt{13}})$
α
α'
$P(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}})$

Figure 3-6b

We now find α' so that $\cos \alpha' = \frac{2}{\sqrt{13}}$ and $\sin \alpha' = \frac{3}{\sqrt{13}} \approx .831$. From our tables we get $\alpha' \approx .98$. Since $\alpha = 2\pi - \alpha'$ we have

$$\alpha \approx 6.28 - .98 \approx 5.30.$$

Therefore, we can write (13) in the convenient form

$$(14) \qquad y = \sqrt{13} \sin (\tfrac{3}{2}t + \alpha), \text{ where } \alpha \approx 5.30.$$

By inspection we can tell that the period is $\frac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}$, the amplitude is $\sqrt{13}$, the phase is 5.30. A sketch appears in Figure 3-6c.

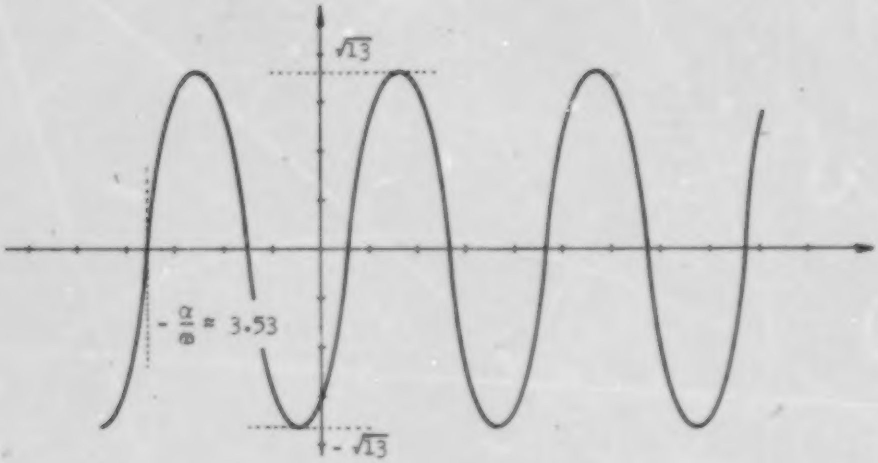


Figure 3-6c

Graph of $y = 2 \sin \frac{3}{2}t - 3 \cos \frac{3}{2}t = \sqrt{13} \sin (\frac{3}{2}t + \alpha)$, where $\alpha \approx 5.30$.

Exercises 3-6

1. Sketch each of the following graphs over at least two of its periods. Show the amplitude and period of each.

   (a) $y = 2 \cos 3t$

   (b) $y = 2 \cos (\frac{3t}{2})$

   (c) $y = 3 \cos (-2t)$

   (d) $y = -2 \sin (\frac{t}{3})$

   (e) $y = -2 \sin (2t + \pi)$

   (f) $y = 5 \cos (3t + \frac{\pi}{6})$

2. Without computing the value of α, find the amplitude and the period of each.

   (a) $y = \sin 3t + \cos 3t$

   (b) $y = -2 \cos \pi t + \sin \pi t$

   (c) $y = 2 \sin \frac{3}{2}t - 2 \cos \frac{3}{2}t$

   (d) $y = 8 \cos \frac{\pi t}{3} + 6 \sin \frac{\pi t}{3}$

   (e) $y = 6 \sin \frac{2t}{5} - \frac{5}{2} \cos \frac{2t}{5}$

   (f) $y = -\sqrt{11} \cos \frac{5\pi t}{4} - 4 \sin \frac{5\pi t}{4}$

3. Express each of the following equations in the form of $y = A \sin (x + \alpha)$, (where $0 \leq \alpha < 2\pi$):

   (a) $y = \sin x + \sqrt{3} \cos x$

   (b) $y = -\sin x + \cos x$

   (c) $y = -\sqrt{3} \sin x - \cos x$

   (d) $y = \sin x - \cos x$

4. Express each equation in Number 3 in the form $y = A \cos (x + \alpha)$, (where $0 \leq \alpha < 2\pi$), by two methods:

   (i) Use the formula for the cosine of the sum of two angles; i.e., $\cos (\theta + \varphi)$.

   (ii) Convert the answers of Number 3 to $A \cos (x + \alpha)$ by the use of trigonometric identities, such as $\sin \varphi = \cos (\frac{\pi}{2} - \varphi)$, $\sin (\theta + 2\pi) = \sin \theta$, $\cos (-\theta) = \cos \theta$, etc.

5. (a) Using the addition formulas, show that $y = 9\sqrt{2} \sin \pi t - 3\sqrt{6} \cos \pi t$ may be put into the form of any one of the following $(0 \leq \alpha < 2\pi)$

   (i) $y = A \sin (\omega t + \alpha)$

   (ii) $y = A \sin (\omega t - \alpha)$

   (iii) $y = A \cos (\omega t - \alpha)$

   (iv) $y = A \cos (\omega t + \alpha)$

   (b) By the use of trigometric identities show that the four expressions of part (a) are equivalent.

   (c) Sketch the graph indicating the period $\frac{2\pi}{\omega}$, the amplitude A, and (using the form $y = A \sin (\omega t + \alpha)$) indicating the phase $\frac{\alpha}{\omega}$.

6. Express each of the following equations in the form $y = A \cos(\pi t - \alpha)$ for some appropriate real numbers A and α.

   (a) $y = 4 \sin \pi t - 3 \cos \pi t$

   (b) $y = -4 \sin \pi t + 3 \cos \pi t$

   (c) $y = -4 \sin \pi t - 3 \cos \pi t$

   (d) $y = 3 \sin \pi t + 4 \cos \pi t$

   (e) $y = 3 \sin \pi t - 4 \cos \pi t$

7. Without actually computing the value of α, show on a diagram how A and α can be determined from the coefficients B and C of $\cos \omega t$ and $\sin \omega t$ if each of the following expressions of the form $B \cos \omega t + C \sin \omega t$ is made equal to $A \cos(\omega t - \alpha)$. Compute α, and find the maximum and minimum values of each expression, and its period. Give reasons for your answers.

   (a) $3 \sin 2t + 4 \cos 2t$

   (b) $2 \sin 3t - 3 \cos 3t$

   (c) $-\sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)$

8. Verify that the superposition of any two pure waves $A \cos(\omega t - \alpha)$ and $B \cos(\omega t - \beta)$ is a pure wave of the same frequency, that is, that there exist real values C and γ such that

$$A \cos(\omega t - \alpha) + B \cos(\omega t - \beta) = C \cos(\omega t - \gamma).$$

*9. Show that any wave of the form

$$y = B \cos(\mu t - \beta), \quad (\mu \neq 0),$$

can be written in the form

$$y = A \cos(\omega t - \alpha)$$

where A is non-negative, ω positive and $0 \leq \alpha < 2\pi$.

3-7. Analysis of General Waves. Period

We have seen that the superposition of two waves, each with period $\frac{2\pi}{\omega}$, such as those given by

$$y = B \cos \omega t \quad \text{and} \quad y = C \sin \omega t$$

gives a pure wave

$$y = A \sin (\omega t + \alpha)$$

of the same period. Now we direct our attention to the superposition of two waves with different periods. Suppose, for example, we had to deal with

$$(1) \qquad y = 2 \sin 3x - 3 \cos 2x.$$

The period of $\sin 3x$ is $\frac{2\pi}{3}$, the period of $\cos 2x$ is $\frac{2\pi}{2} = \pi$. At this point a simple observation is helpful -- namely, if a is a period of f then $2a$, $3a$, $4a$, etc. are also periods of f. For example, we know that

$$\sin 3(x + \frac{2\pi}{3}) = \sin (3x + 2\pi) = \sin 3x$$

and hence it follows that

$$\begin{aligned} \sin 3(x + 2(\frac{2\pi}{3})) &= \sin 3(x + \frac{2\pi}{3} + \frac{2\pi}{3}) \\ &= \sin [3(x + \frac{2\pi}{3}) + 2\pi] \\ &= \sin 3(x + \frac{2\pi}{3}) = \sin 3x. \end{aligned}$$

In general, we must have for each integer n

$$\sin 3(x + n(\frac{2\pi}{3})) = \sin 3x$$

and

$$\cos 2(x + n\pi) = \cos 2x.$$

In particular, we have:

$$\begin{aligned} \sin 3(x + 2\pi) &= \sin 3x \\ \cos 2(x + 2\pi) &= \cos 2x \end{aligned}$$

so that 2π is a period for both $\sin 3x$ and $\cos 2x$. Thus the function defined by (1) has period 2π. The number 2π is the least common multiple of the respective periods $\frac{2\pi}{3}$ and π.

In general, suppose that

$$y = A \sin (at + \alpha) + B \sin (bt + \beta). \tag{2}$$

If $a = b$, we can proceed as in the previous section to express y as a pure wave with period $\frac{2\pi}{a}$. If $a \neq b$, then y may still be periodic but is no longer a pure wave.

Suppose $a \neq b$ but that $\frac{2\pi}{a}$ and $\frac{2\pi}{b}$ have a common multiple, that is, there are positive integers m and n such that

$$m \frac{2\pi}{a} = n \frac{2\pi}{b} . \tag{3}$$

We can then choose m and n so that they have no common factors and (3) holds. In this case, the relation (2) is periodic with period $m \frac{2\pi}{a}$. This is exactly the situation in (1) where $\alpha = 0$, $a = 3$, $\beta = -\frac{\pi}{2}$, $b = 2$ and we can choose $m = 3$, $n = 2$ so that

$$m \frac{2\pi}{a} = n \frac{2\pi}{b} .$$

The period of (1) is then $m \cdot \frac{2\pi}{a} = 3 \cdot \frac{(2\pi)}{3} = 2\pi$.

Of course, it may be that $\frac{2\pi}{a}$ and $\frac{2\pi}{b}$ have no common multiple, in which case (2) is not periodic. For example, the function

$$y = \sin \pi x + \cos x$$

is not periodic. This is difficult to prove and its proof is omitted.

The periodicity of (1) is thus easy to determine. There is little else we can conclude in general about (1). About all we can do to simplify matters is to sketch separately the graphs of

$$u = 2 \sin 3x, \quad v = 3 \cos 2x$$

and $y = u - v$. The result is shown by the three curves in Figure 3-7a.

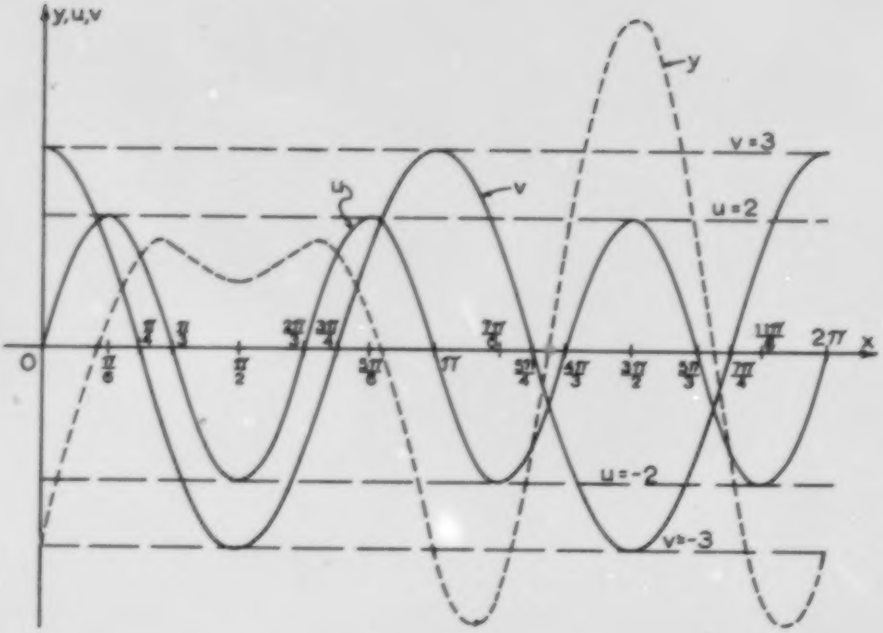


Figure 3-7a

$u = 2 \sin 3x$, $v = 3 \cos 2x$

$y = u - v = 2 \sin 3x - 3 \cos 2x$,

$0 \leq x \leq 2\pi$.

The superposition of sine and cosine waves of different periods can produce quite complicated curves. In fact, with only slight restrictions, any periodic function can be approximated arbitrarily closely as a sum of a finite number of sines and cosines. The subject of harmonic analysis of Fourier series is concerned with approximating periodic functions in this way. The principal theorem, first stated by Fourier, is that a function f of period a can be approximated arbitrarily closely by sines and cosines for each of which some multiple of the fundamental period is a. Specifically,

$$
\begin{aligned}
f(x) \approx A_0 &+ \left(A_1 \cos \frac{2\pi x}{a} + B_1 \sin \frac{2\pi x}{a}\right) \\
&+ \left(A_2 \cos \frac{4\pi x}{a} + B_2 \sin \frac{4\pi x}{a}\right) \\
&+ \cdots \\
&+ \left(A_n \cos \frac{2n\pi x}{a} + B_n \cos \frac{2n\pi x}{a}\right).
\end{aligned}
\tag{1}
$$

and the more terms we use, the better is our approximation.

As an example, consider the function depicted in Figure 3-7b. This function is defined on the interval $-\pi \leq x < \pi$ by

$$(2)\qquad f(x) = \begin{array}{ll} 0, & \text{if } x = -\pi \\ -1, & \text{if } -\pi < x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } 0 < x < \pi. \end{array}$$

For all other values of x we define $f(x)$ by the periodicity condition

$$f(x + 2\pi) = f(x).$$

This function has a particularly simple approximation as a series of the form (1), namely,

$$(3)\qquad \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin(2n-1)x}{2n-1}\right).$$

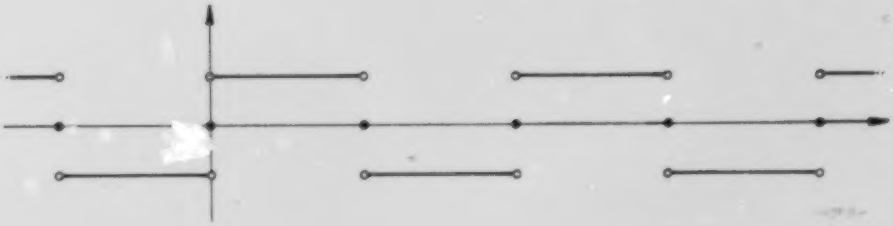


Figure 3-7b

Graph of periodic function.

$$x \to f(x) = \begin{cases} 0, & \text{if } x = -\pi \\ 1, & \text{if } 0 < x < \pi \\ 0, & \text{if } x = 0 \\ -1, & \text{if } -\pi < x < 0 \end{cases} \quad ; \quad f(x + 2\pi) = f(x).$$

Fourier series: $\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin(2n-1)x}{2n-1}\right)$.

As an exercise, you may graph the successive approximations to $f(x)$ by taking one, then two, then three terms of the series, and see how the successive graphs approach the graph of $y = f(x)$.

Exercises 3-7

1. Without sketching, find the periods of the functions defined by the following expressions

   (a) $2 \sin x + \cos 2x$

   (b) $\cos \frac{x}{2} - \sin \frac{x}{3}$

   (c) $\sin \frac{1}{2} \pi x - 3 \cos \pi x$

   (d) $3 \cos 12x - 2 \sin \frac{3}{2} x$

   (e) $1 - 2 \sin^2 x + 2 \sin \frac{x}{2} \cos \frac{x}{2}$

   (f) $|\cos x| + \sin 4x$

   (g) $\sqrt{\cos x} + \sqrt{\sin x}$

   (h) $4 |\sin \frac{3\pi}{2} x| - \frac{1}{3} |\cos 4\pi x|$

2. Sketch the graph of

$$y = 2 \sin x + \sin 2x$$

   by first sketching $y_1 = 2 \sin x$, then $y_2 = \sin 2x$ on the same set of axes for the interval $0 \le x \le 2\pi$.

3. Sketch graphs, for $|x| < \pi$, for each of the following curves.

   (a) $y = \frac{4}{\pi} \sin x$

   (b) $y = \frac{4}{\pi}(\frac{\sin x}{1} + \frac{\sin 3x}{3})$

   (c) $y = \frac{4}{\pi}(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5})$

4. (a) Find the periods of each of the successive terms of the series (3), namely,

$$\sin x, \quad \frac{\sin 3x}{3}, \quad \frac{\sin 5x}{5}, \; \ldots$$

   (b) What terms of the general series (1) are missing? From the symmetry properties of the function f defined by (2) can you see a reason for the absence of certain terms?

5. The symbol $[x]$ stands for the greatest integer less than or equal to the real number x.

   (a) Graph the greatest integer function $x \to [x]$.
   (This function is sometimes called the "integer part" function.)

   (b) Graph the periodic function $x \to x - [x]$.
   Indicate its period and its maximum and minimum values.
   (This function is known as the "fractional part" function.)

   (c) (i) Graph the periodic function $x \to (x)$, where (x) represents the distance from x to the nearest integer. Indicate values of the function.

   (ii) Sketch the graphs of $x \to (2x)$ and $x \to (4x)$.
   On each graph indicate period, maxima, and minima.

   (iii) What is the period of $x \to (nx)$?
   What are its maximum and minimum values?

6. Graph the periodic functions defined below in the interval, $0 \leq t \leq 2$.
   [Note: See No. 5(b) for definition of largest integer function.]

   (a) $f(x) = [\sin \pi x]$
   (b) $f(x) = [\cos \pi x]$
   (c) $f(x) = [2x] - 2[x]$

7. The function f defined by the equation below is periodic. Why?

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

Chapter 4

DERIVATIVES OF CIRCULAR FUNCTIONS

In Chapter 2, we discovered that the derivative of a polynomial function is another polynomial function (of one lower degree), which can be obtained algebraically using the idea of limit. We can show that the derivative of a circular function is another circular function. Using simple geometric arguments we shall show that the derivative of the sine function is the cosine function and the derivative of the cosine function is the negative of the sine function. The first section of this chapter indicates how to obtain these results for the particular cases at $x = 0$, using the same wedge method to find equations of tangent lines employed initially for polynomial graphs.

These results are interpreted in terms of limits of difference quotients in Section 4-2. Later they are extended to pure waves and interpreted in terms of motion. In Section 4-5, the idea of approximating circular functions by polynomials is introduced. The resulting approximation formulas are useful for constructing tables and finding limits.

4-1. The Tangent at the y-Intercept

As was our approach with the polynomial functions, we begin our discussion of the calculus of the circular functions by considering the behavior of their graphs near the y-axis. First we want to find the best straight line approximation to the graph of $y = \cos x$ at the point where $x = 0$. Since $\cos 0 = 1$, we are talking about the point $(0,1)$. We conjecture (from Figure 4-1a) that

(1) the line given by $y = 1$ is tangent to the graph of $y = \cos x$ at the point $(0,1)$.

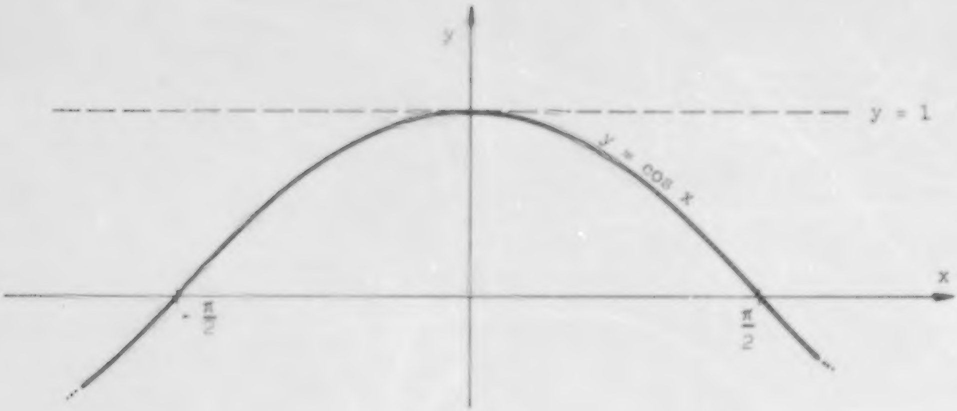


Figure 4-1a

Since the curve is symmetric with respect to the y-axis it is sufficient to consider positive values of x. We know, for $0 < x < \frac{\pi}{2}$, that

(2) $$\cos x < 1$$

and hence the curve lies below the line given by $y = 1$ in the interval $0 < x < \frac{\pi}{2}$. We now wish to show that near the y-axis the curve lies above the line given by

(3) $$y = 1 - \epsilon x$$

where ϵ is some positive number. (See Figure 4-1b.)

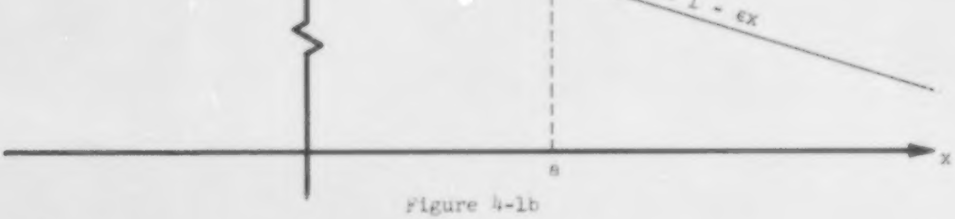


Figure 4-1b

For $x > 0$ and near zero the graph of $y = \cos x$ lies inside the shaded region.

To establish this we use the inequality

(4) $$1 - \cos x \leq \frac{x^2}{2},$$

which we established in Section 3-2. For $x \neq 0$, the quantity x^2 is larger than $\frac{x^2}{2}$; so we have

$$1 - \cos x < x^2, \text{ if } x \neq 0.$$

This can be rewritten as

(5) $$\cos x > 1 - x^2, \text{ if } x \neq 0;$$

that is,

$$\cos x > 1 + (-x)x, \text{ if } x \neq 0.$$

If $0 < x < \epsilon$, then we have

$$1 + (-x)x > 1 - \epsilon x;$$

whence we have

(6) $$\cos x > 1 - \epsilon x \text{ for } 0 < x < \epsilon.$$

In summary, we know that for any positive number ϵ the graph of the cosine function lies inside the wedge of Figure 4-1b on the interval $0 < x < a$ where a is the smaller of ϵ and $\frac{\pi}{2}$. Since we can make ϵ as small as we please, we conclude that the line given by $y = 1$ is indeed the tangent to the graph of $y = \cos x$ at $(0,1)$. Conjecture (1) is established. The line given by $y = 1$ is the best straight line approximation to the cosine curve at the y-axis.

Having established conjecture (1), we now consider the problem of finding the equation of the tangent line to the graph of $y = \sin x$ at $(0,0)$, its point of intersection with the y-axis. We conjecture (from Figure 4-1c) that

(7) the line given by $y = x$ is tangent to the graph of $y = \sin x$ at the point $(0,0)$.

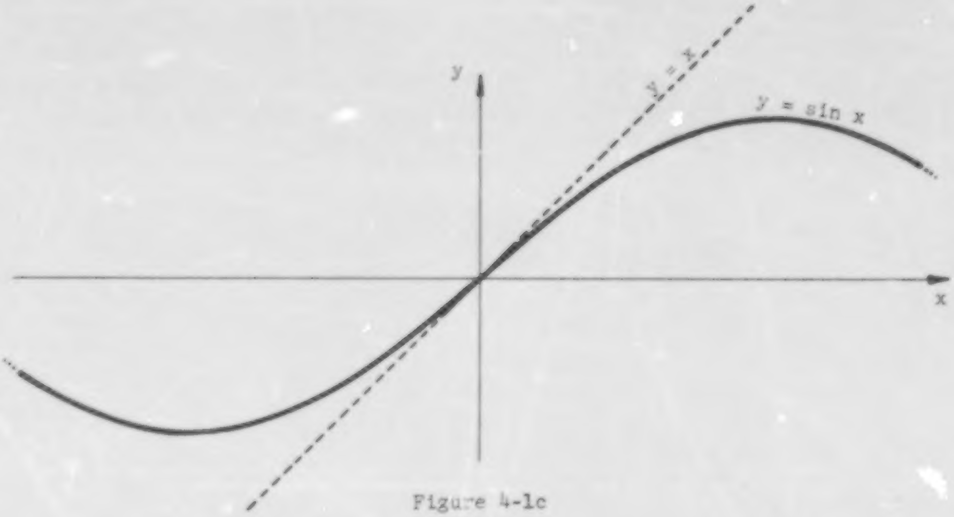


Figure 4-1c

First we shall consider only points to the right of (0,0); in particular we shall restrict our attention to the interval $0 < x < \frac{\pi}{2}$.

To establish our conjecture for x in the first quadrant we shall need two inequalities for the sine function:

$$x(1 - x^2) < \sin x < x.$$

We can derive these inequalities using our unit circle definition of sine. Since, for the moment, we are only concerned with the interval $0 < x < \frac{\pi}{2}$, we picture only the first quadrant of the unit circle in Figure 4-1d.

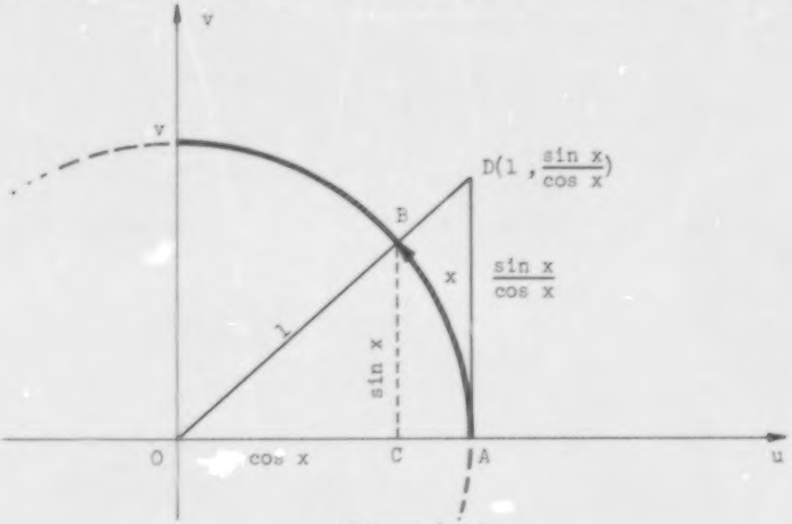


Figure 4-1d

Part of the Unit Circle

The line from B to C is perpendicular to the line from O to A and hence is shorter than the arc AB. If the measure of the length of the arc BA is x, then the measures of the lengths of segments BC and OC are $\sin x$ and $\cos x$, respectively; therefore

$$\sin x < x. \tag{8}$$

This means that, in Figure 4-1c, the graph of the sine function lies below the line given by $y = x$ to the right of $(0,0)$, as we have indicated.

Our second inequality uses an area argument. Referring to Figure 4-1d again, we choose D on the line through O and B so that DA is perpendicular to OA. The first coordinate of D must be 1 (the measure of radius OA). Since $\overleftrightarrow{OB}$ has slope $\frac{\sin x}{\cos x}$, the second coordinate of D must be $\frac{\sin x}{\cos x}$. Thus the region enclosed by triangle OAD has area

$$\frac{1}{2}(OA)(AD) = \frac{1}{2}(1)\left(\frac{\sin x}{\cos x}\right). \tag{9}$$

The area of the circular sector OAB is proportional to x; that is, the area of sector OAB is given by mx, where m is constant. To determine m, we note that if $x = \frac{\pi}{4}$, then the sector OAB is one-eighth of the area of the unit circle; thus

$$\frac{1}{8}(\pi(1)^2) = m\,\frac{\pi}{4}$$

so that $m = \frac{1}{2}$. Therefore, the area of sector OAB is $\frac{1}{2}x$. Since the area of the sector OAB is less than the area of triangle ODA (that is, since the region of sector OAB is within the triangular region ODA), the area of the sector is less than the area of the triangle; that is,

$$\frac{1}{2}x < \frac{1}{2}\,\frac{\sin x}{\cos x}.$$

The cosine function is positive for $0 < x < \frac{\pi}{2}$ so we have

$$\sin x > x \cos x. \tag{10}$$

We know from (5) that if $x \neq 0$ then

$$\cos x > 1 - x^2.$$

We use this in (10) to obtain the inequality

$$\sin x > x(1 - x^2), \quad \text{for } 0 < x < \frac{\pi}{2}. \tag{11}$$

With inequality (11) we can show that $y = x$ is indeed the equation of the tangent to the graph of $y = \sin x$ at $(0,0)$. Suppose, for some positive value ϵ, we have the line whose equation is

$$y = (1 - \epsilon)x.$$

If $x^2 < \epsilon$ and $0 < x < \frac{\pi}{2}$, then we have $1 - x^2 > 1 - \epsilon$. Therefore, we get

$$\sin x > (1 - x^2)x > (1 - \epsilon)x.$$

In summary, the graph of the sine function, to the right of $(0,0)$ and as close as we please to zero (that is, when $x^2 < \epsilon$, and $0 < x < \frac{\pi}{2}$) must lie between the lines given by $y = x$ and $y(1 - \epsilon)x$. (See Figure 4-1e.)

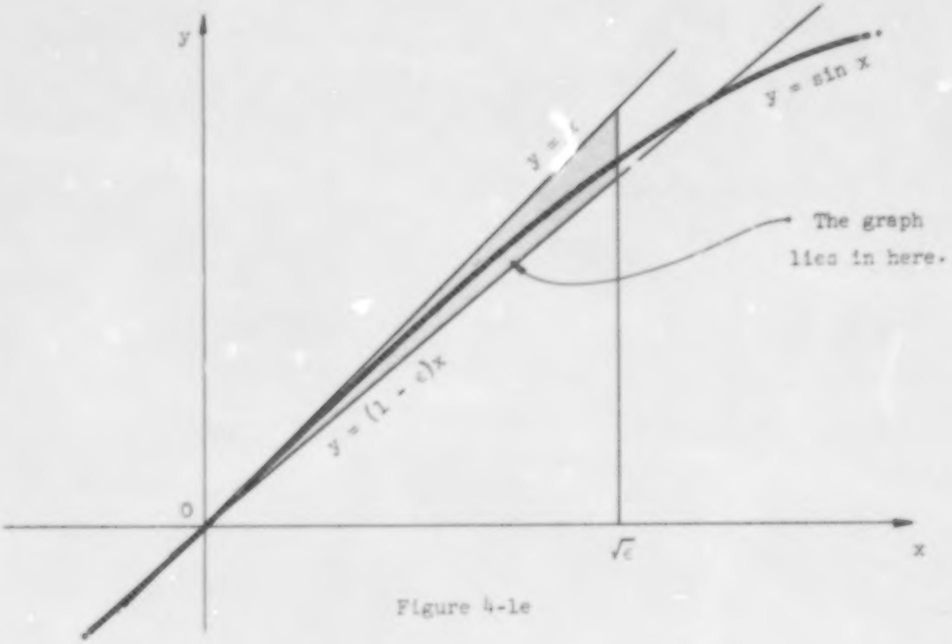


Figure 4-1e

The case when $x < 0$ is now easily handled since the graph of the sine function is symmetric with respect to the origin; that is, $\sin(-x) = -\sin x$, so that when $-\sqrt{\epsilon} < x < 0$ any points $(x, \sin x)$ must lie between the lines given by $y = x$ and $y = (1 - \epsilon)x$. (See Figure 4-1f.)

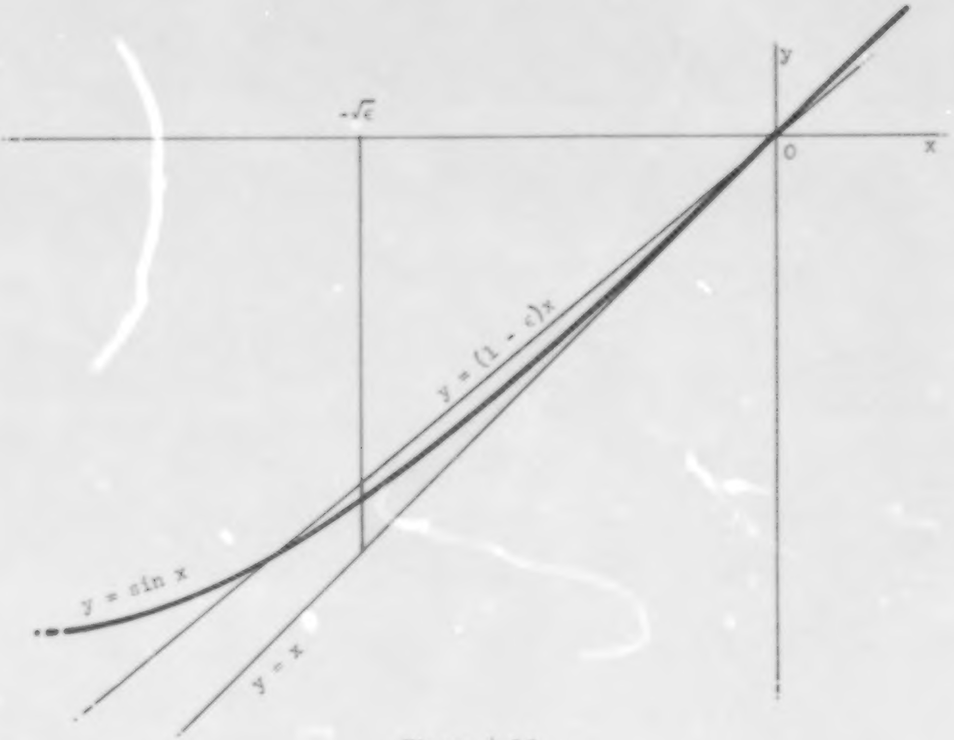


Figure 4-1f

We have now established both conjectures:

(1) the line given by $y = 1$ is the best straight line approximation to the graph of $y = \cos x$ at $(0,1)$;

(2) the line given by $y = x$ is the best straight line approximation to the graph of $y = \sin x$ at $(0,0)$.

Exercises 4-1

1. (a) Write the equation of the tangent to the graph of $y = \cos x$ at $(0,1)$.

   (b) What is the slope of the tangent to the graph of $y = \cos x$ at $(0,1)$?

   (c) Determine $\lim_{h \to 0} \frac{\cos(0+h) - \cos(0)}{h}$.

2. (a) Write the equation of the tangent to the graph of $y = \sin x$ at $(0,0)$.

   (b) What is the slope of the tangent to the graph of $y = \sin x$ at $(0,0)$?

   (c) Determine $\lim_{h \to 0} \frac{\sin(0+h) - \sin(0)}{h}$.

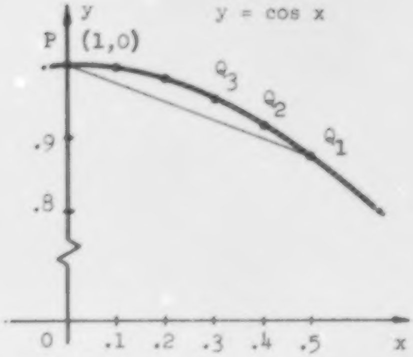
3. Use the results of 1 and 2 to determine

   (a) $\lim_{h \to 0} \frac{\cos h - 1}{h}$,

   (b) $\lim_{h \to 0} \frac{\sin h}{h}$.

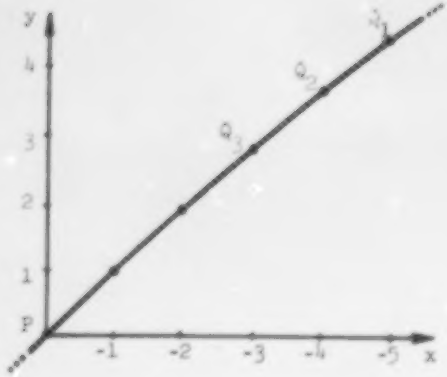
4. (a) To the right is a portion of the graph of $y = \cos x$ near and to the right of $x = 0$. P is the point on the curve where $x = 0$; $Q_i (i = 1, 2, \ldots, 6)$ are points on the curve where $x = .5, .4, .3, .2, .1, .01$. Find the slopes of $PQ_1, PQ_2, \ldots, PQ_6$. Use the table provided.

   (b) Find the equations (in the form $y = b + mx$, where b is y-intercept and m the slope) of each of the lines determined by $PQ_i (i = 1, 2, \ldots, 6)$.



Rad	Cos	Sin
.5	.87758	.47943
.4	.92106	.38942
.3	.95534	.29552
.2	.98007	.09983
.01	.999995	.01000

5. (a) To the right is a portion of the graph of $y = \sin x$ nea. and to the right of $x = 0$. P is the point on the curve where $x = 0$; $Q_i (i = 1, 2, \ldots, 6)$ are points on the curve where $x = .5, .4, .3, .2, .1, .01$. Find the slope of $PQ_1, PQ_2, \ldots, PQ_6$. Use the table provided in Number 1.



(b) Find the equations of each of the lines determined by

$$PQ_i \ (i = 1, 2, \ldots, 6).$$

6. Assuming the following relationships established in this section

(4) $$1 - \cos x \leq \frac{x^2}{2}$$

(5) $$\cos x > 1 - x^2$$

(8) $$\sin x < x$$

(10) $$x \cos x < \sin x$$

show that the following inequalities hold:

(a) $x \cos x < x$

(b) (i) $\frac{1 - \cos x}{x} \leq \frac{x}{2}$ $\quad x > 0$

(ii) $\frac{1 - \cos x}{x} \geq \frac{x}{2}$ $\quad x < 0$

(iii) $\left|\frac{1 - \cos x}{x}\right| \leq \frac{|x|}{2}$ $\quad x \neq 0$

(c) $\sin x > x - \frac{x^3}{2}$ $\quad 0 < x < \frac{\pi}{2}$

(d) $1 - \frac{\sin x}{x} < 1 - \cos x$ $\quad 0 < |x| < \frac{\pi}{2}$

7. Given (from Section 3-2)

$$0 \leq 1 - \cos h \leq \frac{h^2}{2}$$

and (from Section 4-1)

$$h \cos h < \sin h < h, \quad \text{for} \quad 0 < h < \frac{\pi}{2};$$

show that

$$1 - \frac{h^2}{2} < \frac{\sin h}{h} < 1, \quad \text{for} \quad 0 < |h| < \frac{\pi}{2}.$$

8. Using the premises of Number 7, show that

$$\left|\frac{1 - \cos h}{h}\right| < \frac{|h|}{2}, \quad \text{for} \quad h \neq 0.$$

9. Use the results of Numbers 7 and 8 to estimate $\frac{\sin h}{h}$ and $\frac{1 - \cos h}{h}$ for $h = 0.01$ and $h = -0.001$.

10. Use the results of Numbers 7 and 8 to determine

    (a) $\lim\limits_{h \to 0} \frac{\sin h}{h}$

    (b) $\lim\limits_{h \to 0} \frac{1 - \cos h}{h}$

4-2. The Derivative as the Limit of a Difference Quotient

The derivatives of the sine and cosine functions are, respectively, the cosine and the negative of the sine functions. These general results can be obtained by first finding the equation of tangent lines to the graphs of the sine and cosine functions and then discovering the slope functions (derivatives). We shall, however, obtain these derivatives directly as limits of certain difference quotients.

First we shall show that for the sine function

$$f : x \rightarrow \sin x,$$

we can obtain the derivative

$$f': x \rightarrow \cos x$$

by considering the definition of f' given by

(1) $f'(x) =$ the limit of $\dfrac{\sin (x + h) - \sin x}{h}$ as h approaches zero.

The geometric interpretation is that for some fixed point $(x,f(x))$ on the graph of f, the difference quotient

$$\frac{\sin (x + h) - \sin x}{h}$$

is the slope of the line passing through the points

$$(x + h , \sin (x + h)) \text{ and } (x , \sin x).$$

(See Figure 4-2a, where h is shown negative and rather large.)

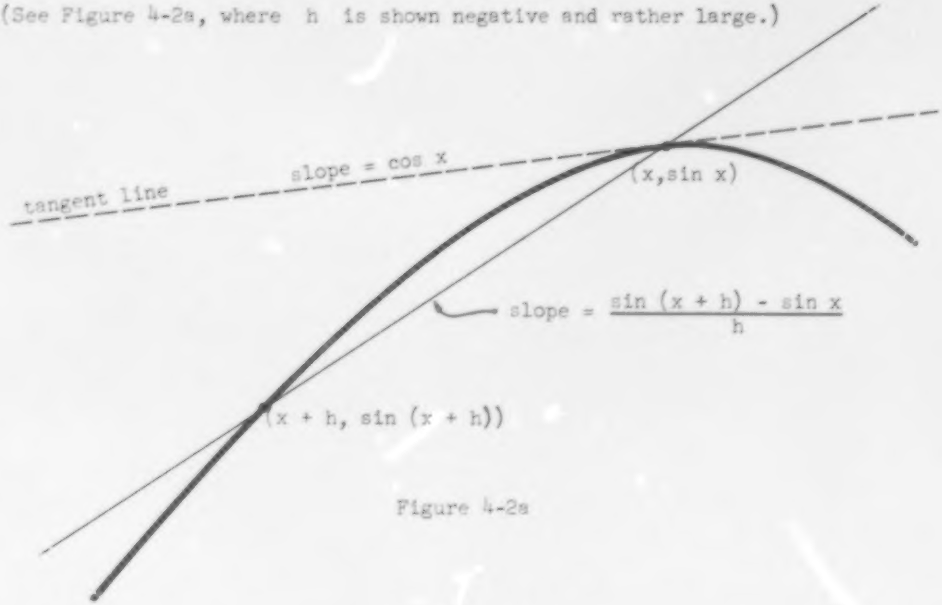


Figure 4-2a

We wish to show that

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x. \qquad (2)$$

This can be shown by arguing (as we did in Section 4-1 for $x = 0$) that the graph of the sine function lies inside any given wedge about the tangent when $|h|$ is small enough. We shall use a more direct method in which we employ the results of Section 4-1. We begin by using the addition formula (6) of Section 3-5. We can write

$$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x \frac{\sin h}{h}.$$

We must now show that as h approaches zero, the first term approaches zero and the second term approaches $\cos x$. In other words we must show that

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

In Section 4-1 we showed that the horizontal line given by $y = 1$ is the best straight line approximation to the graph of $y = \cos x$ at $(0,1)$. Since the slope of the best straight line approximation (horizontal tangent) to the graph of $y = \cos x$ of $x = 0$ is zero, we have

$$\lim_{h \to 0} \frac{\cos(0+h) - \cos(0)}{h} = 0.$$

Therefore,

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

In Section 4-1 we also showed that the line given by $y = x$ is the best straight line approximation to the graph of $y = \sin x$ at $(0,0)$. Since the slope of the tangent to the graph of $y = \sin x$ at $x = 0$ is one, we have

$$\lim_{h \to 0} \frac{\sin(0+h) - \sin 0}{h} = 1.$$

Therefore,

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$

Now we can conclude that

$$\lim_{h \to 0} \frac{\sin (x + h) - \sin x}{h} = \cos x. \tag{2}$$

We have established that

$$D(\sin x) = \cos x. \tag{3}$$

The derivative of the cosine function at any point $(x, \cos x)$ can be obtained by calculating the limit of the difference quotient

$$\frac{\cos (x + h) - \cos x}{h}$$

as h approaches zero. We leave it to the exercises (Exercises 4-2, No. 1) to show that

$$D(\cos x) = -\sin x.$$

Example 4-2a. Find the limit of $\frac{\sin 2h}{h}$ as h approaches zero; that is, evaluate

$$\lim_{h \to 0} \frac{\sin 2h}{h} .$$

We can argue that since

$$\frac{\sin h}{h} \approx 1 \quad \text{for small} \quad |h|,$$

it follows that

$$\frac{\sin 2h}{2h} \approx 1 \quad \text{if} \quad |2h| \quad \text{is small.}$$

We can write

$$\frac{\sin 2h}{h} = 2 \, \frac{\sin 2h}{2h} .$$

If $|h|$ is so small that $|2h|$ is small, then

$$\frac{\sin 2h}{h} \approx 2.$$

Alternatively, and more directly, we can say

$$\begin{aligned} \lim_{h \to 0} \frac{\sin 2h}{h} &= 2 \lim_{h \to 0} \frac{\sin 2h}{2h} \\ &= 2(1) \\ &= 2. \end{aligned}$$

Example 4-2b. Find the equation of the tangent line (best straight line approximation) to the graph of the sine function at the point $(a, \sin a)$, when $a = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{6}$.

We can write $x = a + (x - a)$ so that $\sin x = \sin [a + (x - a)]$. Using addition formula (6) of Section 3-5 we get

$$\sin x = \sin a \cos (x - a) + \cos a \sin (x - a).$$

If we let x approach a then $x - a$ approaches zero. We can replace $\cos (x - a)$ and $\sin (x - a)$ by their best linear approximations: $\cos (x - a)$ by 1, and $\sin (x - a)$ by $x - a$ (from (1) and (7) of Section 4-1). Therefore, as we let x approach a, the expression which gives the best linear approximation to $\sin x$ is

$$\sin a(1) + \cos a(x - a);$$

that is, the equation of the tangent line at the point $(a, \sin a)$ is

$$y = \sin a + \cos a(x - a).$$

When $a = \frac{\pi}{4}$ the equation of the tangent to the sine curve is

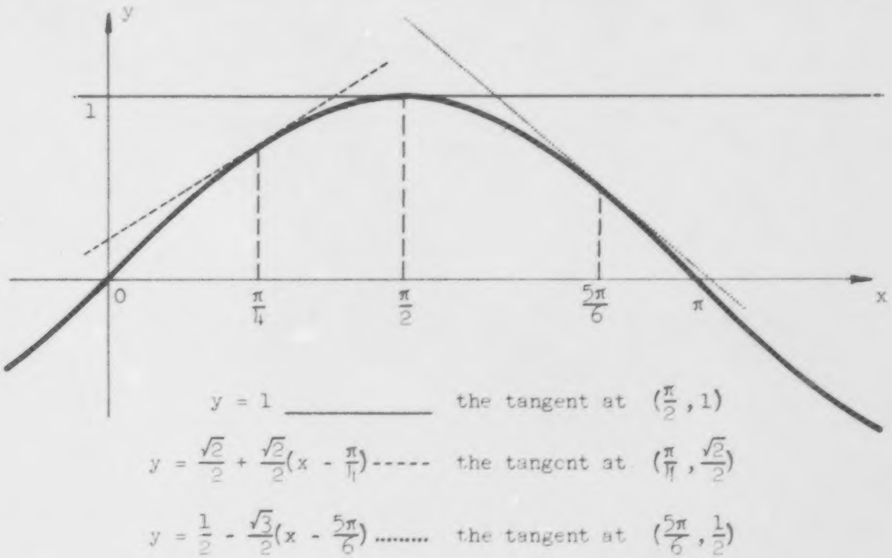
$$y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}).$$

When $a = \frac{\pi}{2}$ the equation of the tangent is

$$y = 1;$$

that is, the tangent to the sine curve at $(\frac{\pi}{2}, 1)$ is horizontal. When $a = \frac{5\pi}{6}$, the equation of the tangent is

$$y = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{5\pi}{6}).$$



$y = 1$ ________ the tangent at $(\frac{\pi}{2}, 1)$

$y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4})$ - - - - - the tangent at $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$

$y = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{5\pi}{6})$ .... the tangent at $(\frac{5\pi}{6}, \frac{1}{2})$

Figure 4-2a

Tangent lines to the graph of $y = \sin x$.

Note that the slopes of these three tangent lines are $\frac{\sqrt{2}}{2}$, 0, and $-\frac{\sqrt{3}}{2}$ respectively. As we should expect, these values of the derivative could have been obtained by direct substitution in (3).

Exercises 4-2

1. (a) Use addition formula (4) of Section 3-5 to show that the difference quotient

$$\frac{\cos (x + h) - \cos x}{h}$$

can be written in the form

$$\cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right).$$

(b) Show that $D(\cos x) = -\sin x$.

2. (a) Assume (from Section 3-5) that

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$
$$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$$

Let $\alpha + \beta = x + h$ and $\alpha - \beta = x$.
Show that

$$\sin (x + h) - \sin x = 2 \cos \left(x + \frac{h}{2}\right) \sin \left(\frac{h}{2}\right).$$

(b) Use part (a) to show that $D(\sin x) = \cos x$.

3. (a) Given (from Section 3-5):

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$
$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta;$$

Prove: $\cos (x + h) - \cos x = -2 \sin \left(x + \frac{h}{2}\right) \sin \left(\frac{h}{2}\right)$

(b) Use part (a) to show that $D(\cos x) = -\sin x$.

4. (a) From the inequality $.99995 < \frac{\sin (.01)}{.01} < 1$, estimate $\sin (.01)$.

(b) From the inequality $-.0005 < 1 - \frac{\cos (-.001)}{(-.001)} < .0005$, estimate $\cos (-.001)$.

5. (a) Using the inequality $1 - \frac{h^2}{2} < \frac{\sin h}{h} < 1$, estimate

(i) $\sin (.1)$
(ii) $\sin (.01)$
(iii) $\sin (.001)$
(iv) $\sin (.0001)$

(b) Using the inequality, $1 - \frac{h^2}{2} \leq \cos h \leq 1$, estimate

(i) $\cos (.1)$
(ii) $\cos (.01)$
(iii) $\cos (.001)$
(iv) $\cos (.0001)$

6. (a) Using $\left|\frac{1 - \cos h}{h}\right| \leq \frac{|h|}{2}$, $h \neq 0$, and the results of Number 5(b) find the value of $\left|\frac{1 - \cos h}{h}\right|$ for

(i) $h = .1$; $h = -.1$ (iii) $h = .001$; $h = -.001$
(ii) $h = .01$; $h = -.01$ (iv) $h = .0001$; $h = -.0001$

(b) By the use of the inequality $1 - \frac{h^2}{2} < \frac{\sin h}{h} < 1$, illustrate that the difference between $\frac{\sin h}{h}$ and 1 becomes smaller and smaller as the values assigned to $|h|$ decrease.
Show this for the following values of h:

(i) $h = .1$; $h = -.1$ (iii) $h = .001$; $h = -.001$
(ii) $h = .01$; $h = -.01$ (iv) $h = .0001$; $h = .0001$

7. The limit of $\frac{\sin h}{h}$ as h tends to zero is one. This may be stated symbolically as $\lim\limits_{h \to 0} \frac{\sin h}{h} = 1$. Find the $\lim\limits_{x \to 0} \frac{\sin 3x}{2x}$ by two methods:

(a) by use of the inequality $1 - \frac{h^2}{2} < \frac{\sin h}{h} < 1$

(b) by direct application of $\lim\limits_{h \to 0} \frac{\sin h}{h} = 1$

8. When appropriate use

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{1 - \cos h}{h} = 0,$$

in evaluating the following:

(a) $\lim\limits_{x \to 0} \frac{\sin x}{3x}$

(b) $\lim\limits_{x \to 0} \frac{\sin^2 x}{x}$

(c) $\lim\limits_{x \to 0} \frac{\tan x}{x}$

(d) $\lim\limits_{h \to 0} \frac{h}{\sin h}$

(e) $\lim\limits_{x \to 0} \frac{\sin 7x}{\sin 3x}$

(f) $\lim\limits_{x \to 0} \frac{\cos 7x}{\cos 3x}$

(g) $\lim\limits_{\theta \to 0} \frac{1 - \cos \theta}{\theta^2}$

(h) $\lim\limits_{x \to 0} \frac{\tan 2x}{\sin x}$

(i) $\lim\limits_{t \to 0} \frac{\sin 2t}{2t^2 + t}$

(j) $\lim\limits_{\theta \to 0} \theta \cot 2\theta$

(k) $\lim\limits_{\theta \to \frac{\pi}{2}} \frac{\cos \theta}{\frac{\pi}{2} - \theta}$

(ℓ) $\lim\limits_{x \to 0} \frac{\sin 5x - \sin 3x}{x}$

9. (a) Evaluate $\lim_{x \to a} \frac{\sin x - \sin a}{x - a}$.

(b) Use the result of part (a) to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

10. (a) Find the slope of the line tangent to the graph of $y = \sin x$ at the point where

(i) $x = \frac{\pi}{3}$

(ii) $x = \frac{\pi}{6}$

(iii) $x = \frac{3\pi}{4}$

(iv) $x = \pi$

(v) $x = \frac{3\pi}{2}$

(vi) $x = 0$

(b) Write an equation of the line tangent to the graph of $y = \sin x$ at each of the points in part (a).

11. (a) Find the slope of the line tangent to the graph of $y = \cos x$ at the point where

(i) $x = \frac{\pi}{6}$

(ii) $x = \frac{2\pi}{3}$

(iii) $x = -\frac{\pi}{4}$

(iv) $x = 0$

(b) Write an equation of the line tangent to the graph of $y = \cos x$ at each of the points in part (a).

12. (a) For what values of x does the graph of $y = \sin x$ have a horizontal tangent line?

(b) For what value of x does the graph of $y = \cos x$ have a horizontal tangent line?

13. (a) For what values of x does the graph of $y = \sin x$ have the tangent line given by $y = x$ or a line parallel to the line given by $y = x$?

(b) Answer the above question for the graph of $y = \cos x$.

14. (a) For what values of x does the graph of $y = \sin x$ have the tangent line given by $y = -x$ or a line parallel to the line given by $y = -x$?

(b) Answer the above question for the graph of $y = \cos x$.

15. (a) For what value of x does the graph of $y = \sin x$ have the tangent line whose equation is $2y = x$ or parallel to that line?

(b) Answer the above question for the graph of $y = \cos x$.

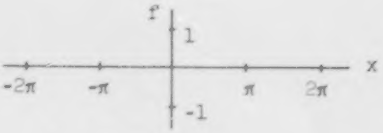
16. (a) If $f : x \to \sin x$, then $f' : x \to \cos x$. Find $f'(600\pi)$, $f'(-200\pi - \frac{\pi}{6})$, $f'(60\pi - \frac{5\pi}{4})$.

(b) If $g : x \to \cos x$, then $g' : x \to -\sin x$. Find $g'(600\pi)$, $g'(-200\pi - \frac{\pi}{6})$, $g'(60\pi - \frac{5\pi}{4})$.

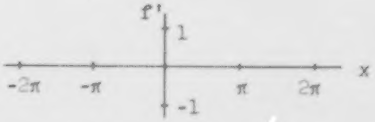
17. (a) If $f : x \to \sin x$ and $f' = h$ show that $h' = -f$.

(b) If $f : x \to \sin x$, $g : x \to \cos x$ and $f' = h$, $g' = j$, show that $h' = j$ and $h = -j'$.

18. (a) In the interval $|x| \leq 2\pi$ for what values of x does the function, $f : x \to \cos x$, increase? For what values does the function decrease?



(b) Sketch f and f' (on different graphs but using the same scale) to illustrate your answer.



19. Without finding its derivative, determine, for $|x| \leq \pi$, for which values of x the function $f : x \to \cos 2x$ increases and for which this function decreases by the following two procedures:

(a) Extend the solution of part (a) of Number 1 to the present problem.

(b) Inspect the graph of the function

$$f : x \to \cos 2x.$$

20. In the interval $|x| \leq \pi$, determine by the following two procedures the values of x for which the function $f : x \to \cos(2x + \frac{\pi}{2})$ increases and for which it decreases.

(a) Extend the solution of part (a) of Number 1.

(b) Inspect the graph of the function,

$$f : x \to \cos(2x + \frac{\pi}{2})$$

21. For what values of x over the interval $0 \leq x \leq 2\pi$ do the sine function and the cosine function both increase? For what values of x do both functions decrease?

22. For what values of x over the interval $0 \leq x \leq 2\pi$ do the following functions all increase? all decrease?

    (a) $x \rightarrow \sin \frac{x}{2}$

    (b) $x \rightarrow \sin x$

    (c) $x \rightarrow \sin 2x$

23. How large must the constant a be for the following functions to be increasing?

    (a) $f : x \rightarrow ax - \sin x$

    (b) $f : x \rightarrow ax + \cos x$

24. (a) Demonstrate the concavity (downward flexure) of the graph of $x \rightarrow \sin x$ at $x = .4$ by showing that at $x = .3$ and at $x = .5$ the curve lies below the tangent to the curve at $x = .4$.

    (b) Demonstrate the convexity (upward flexure) of the graph of $x \rightarrow \sin x$ at $x = -.4$ by showing that at $x = -.3$ and at $x = -.5$ the curve lies above the tangent to the curve at $x = -.4$.

4-3. Linear Substitution

Now we turn to the problem of determining the derivative of a function specified by an equation of the type

$$y = k \sin (ax + b).$$

Such a function is obtained from $x \rightarrow y = \sin x$ by linear substitution; that is, x is replaced by the linear expression $ax + b$, and y by $\frac{y}{k}$.

To determine $D[k \sin (ax + b)]$, we try to obtain the limit, as h approaches zero, of the difference quotient

$$\frac{k \sin [a(x + h) + b] - k \sin (ax + b)}{h}$$

which can be written as

$$\frac{k \sin [(ax + b) + ah] - k \sin (ax + b)}{h}.$$

Using addition formula (6) from Section 3-5, we can express the numerator of the difference quotient as

$$k \sin (ax + b) \cos (ah) + k \cos (ax + b) \sin (ah) - k \sin (ax + b).$$

Factoring, we obtain for the numerator

$$k \sin (ax + b)[\cos (ah) - 1] + k \cos (ax + b) \sin (ah).$$

The difference quotient can be expressed in the more useful form

$$k \sin (ax + b)[\frac{\cos (ah) - 1}{h}] + k \cos (ax + b)[\frac{\sin (ah)}{h}].$$

As we prepare to take the limit as h approaches zero we make one further change (suggested by Example 4-2a and Exercises 4-2, No. 8) in the form of the difference quotient:

$$ak \sin (ax + b)[\frac{\cos (ah) - 1}{(ah)}] + ak \cos (ax + b)[\frac{\sin (ah)}{(ah)}].$$

As h approaches zero, ah approaches zero; and we know (from (1) and (7) of Section 4-1) that $\frac{\cos (ah) - 1}{(ah)}$ approaches zero and $\frac{\sin (ah)}{(ah)}$ approaches one. We conclude that

$$\begin{aligned} D[k \sin(ax + b)] &= ak \sin(ax + b)\left[\lim_{h \to 0} \frac{\cos(ah - 1)}{(ah)}\right] + ak \cos(ax + b)\left[\lim_{h \to 0} \frac{\sin(ah)}{(ah)}\right] \\ &= ak \sin(ax + b) \cdot 0 + ak \cos(ax + b) \cdot 1 \\ &= ak \cos(ax + b). \end{aligned}$$

We can express our result in the form

(1) $$\text{if} \quad f : x \to k \sin (ax + b),$$
$$\text{then} \quad f': x \to ak \cos (ax + b).$$

Similarly we could show that

(2) $$\text{if} \quad g : x \to k \cos (ax + b),$$
$$\text{then} \quad g': x \to -ak \sin (ax + b).$$

Using (1) and (2), we can obtain the slope of a tangent to any sinusoidal curve directly.

Example 4-3a. If $f : x \to 3 \sin 2x$, find $f'(\pi)$. From (1) we have

$$f': x \to 3(2 \cos 2x) = 6 \cos 2x$$

so that

$$f'(\pi) = 6 \cos 2\pi = 6.$$

Example 4-3b. Find the equation of the tangent line to the graph of $y = \cos (x + \frac{\pi}{4})$ at the point $(\frac{\pi}{12}, \frac{1}{2})$. Using y' to denote the value of the derivative, we use (2) to obtain

$$y' = -\sin (x + \frac{\pi}{4}).$$

At the point $(\frac{\pi}{12}, \frac{1}{2})$ the slope of the tangent to the graph is $-\sin (\frac{\pi}{12} + \frac{\pi}{4}) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$. The equation of the tangent is

$$y = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{\pi}{12}).$$

We can use the graphs of $y = \cos (x + \frac{\pi}{4})$ and $y = \cos x$ to give one geometrical interpretation of this result. (See Exercises 4-3, No. 8.)

Example 4-3c. Use (1) and the facts (from Section 3-2) that

(3) $$\cos x = \sin (\frac{\pi}{2} - x) \quad \text{and} \quad \sin x = \cos (\frac{\pi}{2} - x)$$

to obtain the derivative of the cosine function.

We have from (3)

$$\cos: \; x \to \sin (\frac{\pi}{2} - x).$$

We use (1) with $k = 1$, $a = -1$, and $b = \frac{\pi}{2}$ to obtain the derivative

$$\cos': \ x \to (-1) \cos (\tfrac{\pi}{2} - x) = -\sin x.$$

Example 4-3d. Find the derivative of

$$f : x \to \sin x^{\circ}.$$

The function f, sometimes written

$$f : x \to \sin^{\circ} x,$$

is the function whose value at x is the sine of x degrees. Since

$$x^{\circ} \to \frac{2\pi}{360} x \ \text{ radians},$$

we can express f as

$$f : x \to \sin \frac{2\pi}{360} x.$$

Thus, formula (1) gives

$$f': x \to \frac{2\pi}{360} \cos (\frac{2\pi}{360} x).$$

The primary justification for defining $\sin x$ in terms of arc length x, rather than degree measure, is that we then obtain the simple formula

$$D \sin x = \cos x$$

If we had used degree measure then we would have had the less satisfactory formula

$$D \sin x = \frac{2\pi}{360} \cos x.$$

Exercises 4-3

1. Using formulas (1) and (2), find the following.

   (a) $D[\sin (3x - \frac{\pi}{4})]$ (d) $D[5 \cos (\frac{\pi}{6} - \frac{x}{3})]$

   (b) $D[-2 \cos (\frac{x}{2} + \frac{2\pi}{3})]$ (e) $D[\frac{1}{2} \sin (-x + \frac{\pi}{2})]$

   (c) $D[-\sin (\frac{\pi}{3} - x)]$ (f) $D[-\pi \cos (2\pi - 2x)]$

2. For the following functions determine

   (a) the slope of the graph at the point indicated for each:

   (i) $f : x \rightarrow \cos (-x + \frac{\pi}{3})$, $x = -\frac{7\pi}{6}$,

   (ii) $f : x \rightarrow -\sin (2x - \frac{\pi}{6})$, $x = \frac{\pi}{6}$,

   (iii) $f : x \rightarrow 3 \cos (\frac{\pi}{4} + 2x)$, $x = \frac{\pi}{2}$,

   (iv) $f : x \rightarrow \frac{1}{2} \sin (\frac{3\pi}{2} - \frac{x}{2})$, $x = 0$;

   (b) the equation of the line tangent to the graph of each function in part (a) at the point indicated.

3. If two functions f and g are directly proportional, we mean that there is a constant c such that $g(x) = cf(x)$.

   (a) How are their derivatives related?

   (b) Illustrate your answer to part (a) when $f(x) = \sin 2x$, and $g(x) = -2 \cos (2x - \frac{\pi}{2})$.

4. (a) (i) For what value(s) of x over the interval $0 \leq x < 2\pi$ does the graph of the function

   $$f : x \rightarrow \sin x + \cos x$$

   have a horizontal tangent line?

   (ii) Give the equation(s) of the tangent line(s).

   (b) Answer the questions of part (a) for the function

   $$g : x \rightarrow 4 \sin \frac{x}{2} + 4 \cos \frac{x}{2} .$$

   (c) Answer the questions of (a) for the function

   $$h : x \rightarrow 3 \sin (2x + \frac{\pi}{4}) + 3 \cos (2x + \frac{\pi}{4}).$$

5. Show that if

$$f : x \to k \cos (ax + b)$$

then

$$f' : x \to -ka \sin (ax + b).$$

Employ the method of linear substitution used for the function

$$x \to k \sin (ax + b)$$

at the beginning of this section.

6. In Example 4-3c we used the facts that

$$\cos x = \sin \left(\frac{\pi}{2} - x\right) \quad \text{and} \quad \sin x = \cos \left(\frac{\pi}{2} - x\right)$$

to obtain the derivative of $x \to \cos x$. Use the same method to find the derivative of

$$x \to k \cos (ax + b),$$

given that $D[k \sin (ax + b)] = ka \cos (ax + b)$.

7. (a) Find the limit of $\dfrac{\sin \alpha^\circ}{\alpha^\circ}$ as α° approaches zero.

   (b) Find the derivative of

$$f : x \to \cos x^\circ.$$

8. (a) On one set of coordinate axes sketch the graphs of $f : x \to \cos x$ and $g : x \to \cos \left(x + \frac{\pi}{4}\right)$.

   (b) Show that the tangent line to the graph of f at the point $\left(\frac{\pi}{3}, f\left(\frac{\pi}{3}\right)\right)$ is parallel to the tangent line to the graph of g at the point $\left(\frac{\pi}{12}, g\left(\frac{\pi}{12}\right)\right)$.

9. Use the difference quotient definition of derivative as it applies to the sine and cosine functions to evaluate each of the following

   (a) $\lim\limits_{h \to 0} \dfrac{\sin (x + h) - \sin x}{h}$

   (b) $\lim\limits_{h \to 0} \dfrac{\cos x - \cos (x + h)}{h}$

   (c) $\lim\limits_{h \to 0} \dfrac{\cos (3x + h) - \cos 3x}{h}$

(d) $\lim\limits_{h \to 0} \dfrac{-3 \sin (\frac{x}{2} + \frac{\pi}{4} + h) + 3 \sin (\frac{x}{2} + \frac{\pi}{4})}{\frac{x}{2} + \frac{\pi}{4}}$

(e) $\lim\limits_{k \to 0} \dfrac{\sin (\pi + k) - \sin (\pi)}{k}$

(f) $\lim \dfrac{-\cos (\frac{\pi}{4} + h) + \cos \frac{\pi}{4}}{2h}$

10. Sketch the graph of each of the following on the interval $[-\pi, \pi]$.

(a) $y = \sin x$

(b) $y = \sin (x - \frac{\pi}{2})$

(c) $y = -\sin x$

(d) $y = \sin (-x)$

(e) $y = \sin (\frac{\pi}{2} - x)$

(f) $y = \cos x$

(g) $y = \cos (x - \frac{\pi}{2})$

(h) $y = \cos (-x)$

(i) $y = \cos (\frac{\pi}{2} - x)$

11. Find $\frac{dy}{dx}$ for each of the following.

(a) $y = \sin x$

(b) $y = \sin (x - \frac{\pi}{2})$

(c) $y = -\sin x$

(d) $y = \sin (-x)$

(e) $y = \sin (\frac{\pi}{2} - x)$

(f) $y = \cos x$

(g) $y = \cos (x - \frac{\pi}{2})$

(h) $y = \cos (-x)$

(i) $y = \cos (\frac{\pi}{2} - x)$

4-4. Velocity and Acceleration

In Section 3-4 we discussed motion in a circle with radius 1. (See Figure 4-4a.) If point P is moving counterclockwise starting at $A(1,0)$ when $t = 0$ and if the arc is covered at the uniform rate of 1 unit per second, the coordinates of P at time t are

$$x = \cos t$$
$$y = \sin t.$$

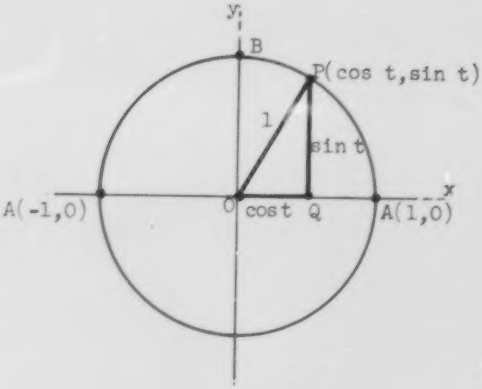


Figure 4-4a

Let us draw a line through P perpendicular to the x-axis. Let Q be the foot of the perpendicular. We shall study the motion of the point Q as t increases. When $t = \frac{\pi}{2}$ (a quarter circumference), P has reached B on the y-axis and Q has reached O. As t increases from $\frac{\pi}{2}$ to π, Q moves from O to $A'(-1,0)$. From $t = \pi$ to $t = \frac{3\pi}{2}$, Q moves from A' back to O. At $t = 2\pi$, Q is back at A. As we see, Q oscillates back and forth on the x-axis. This motion repeats at time intervals of 2π.

Let us find the velocity of the point Q at any time t. To do this we take the derivative of the function

$$f : t \rightarrow \cos t, \tag{1}$$

which represents the position of the point Q at time t. As we know, the derivative of f is

$$f' : t \rightarrow -\sin t. \tag{2}$$

During the first half-revolution $(0 < t < \pi)$, the velocity is negative which shows that Q is moving to the left. The velocity is 0 at $t = 0$ and $t = \pi$. At $t = \frac{\pi}{2}$ when $Q = 0$, $f'(\frac{\pi}{2}) = -1$. The speed, which is the absolute value of the velocity, has its maximum value when $t = \frac{\pi}{2}$.

During the second half-revolution $(\pi < t < 2\pi)$, Q is moving to the right. $f'(\frac{3\pi}{2}) = 1$ is the maximum velocity and speed during this time interval. The motion repeats after $t = 2\pi$.

What is the acceleration of Q at any time t? The acceleration is the derivative of the velocity f', that is, the second derivative f'' of f. In this case, since

(2) $$f': t \to -\sin t,$$

(3) $$f'': t \to -\cos t.$$

The velocity decreases when the acceleration is negative and increases when the acceleration is positive. Let us see how this works out. During the first quarter-revolution, f'' is negative but f' is also negative. A decrease of f' means an increase in its absolute value, that is, an increase in speed.

Between $t = \frac{\pi}{2}$ and $t = \pi$, $\cos t$ is negative and $\sin t$ is positive. Hence, f' is negative and f'' is positive; f' increases from -1 to 0 and the speed decreases from 1 to 0.

Between $t = \pi$ and $t = \frac{3\pi}{2}$, $\sin t$ and $\cos t$ are both negative so that both f' and f'' are positive. Both velocity and speed increase from 0 to 1.

Finally, for $\frac{3\pi}{2} < t < 2\pi$, f'' is negative and f' is positive so that Q slows down again to 0.

Example 4-4a. Show that $f''(t) = -f(t)$. In what quadrants is f''

(a) positive?

(b) negative?

Since $$f''(t) = -\cos t$$

and $$f(t) = \cos t$$

$$f''(t) = -f(t)$$

(more briefly, $f'' = -f$).

Hence, $f''(t)$ is positive when $f(t) = x$ is negative, that is, in quadrants II and III; f'' is negative when x is positive, that is, in quadrants I and IV.

Example 4-4b. To find the position, velocity and acceleration of Q when $t = \frac{\pi}{3}$. We have

$$f(\tfrac{\pi}{3}) = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$f'(\tfrac{\pi}{3}) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2} \approx -0.86$$

$$f''(\tfrac{\pi}{3}) = -\cos\frac{\pi}{3} = -\frac{1}{2}.$$

Hence, after $\frac{\pi}{3}$ seconds, the point Q is at $x = \frac{1}{2}$ and is moving to the left at $\frac{\sqrt{3}}{2}$ distance units per second. The acceleration is $-\frac{1}{2}$ distance units per second each second, which means that the velocity is decreasing at the rate of $\frac{1}{2}$ unit per second per second. (The speed is increasing at this rate.)

We can generalize our discussion by considering uniform motion on a circle of radius r where the point P moves the distance kt in t seconds. (See Figure 4-4b.) Let θ be the measure of angle AOP in radians. Then

$$\theta = \frac{kt}{r}.$$

As in Section 3-4 we let $\frac{k}{r} = \omega$ so that

$$\theta = \omega t.$$

The coordinates of P are

$$x = r\cos\omega t$$
$$y = r\sin\omega t.$$

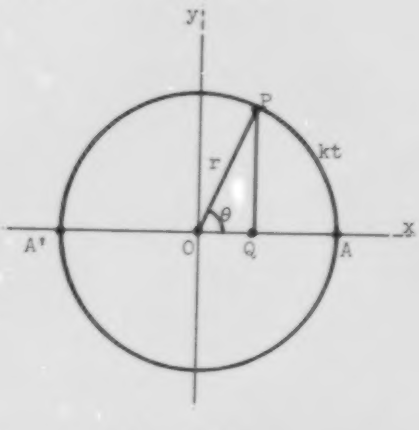


Figure 4-4b

We let

(4) $$f : t \to r\cos\omega t.$$

If Q is the projection of P on the x-axis, then the velocity and acceleration of Q are given by the functions

(5) $$f' : t \to -\omega r\sin\omega t$$

and

(6) $$f'' : t \to -\omega^2 r\cos\omega t$$

respectively. Now Q oscillates between $A(r,0)$ and $A'(-r,0)$. The time required for one oscillation back and forth is obtained by using the circumference $2\pi r$ as the arc, so that $kt = 2\pi r$. We call this time T the *period* of the motion. Then

$$T = \frac{2\pi r}{k} = \frac{2\pi}{\omega} .$$

Note that

$$f'' = -\omega^2 f \tag{7}$$

which is called a *differential equation*. We say that $f(x) = r \cos \omega t$ is a *solution* of this equation.

Example 4-4c. A point P moves on a circle of radius 10 feet at the rate of 2 feet per second. Find the position of its projection Q and the velocity and acceleration of Q after 5 seconds. Since $\omega = \frac{k}{r} = \frac{2}{10} = \frac{1}{5}$,

$$f : t \rightarrow 10 \cos \frac{t}{5} ,$$

$$f' : t \rightarrow 10 \cdot \frac{1}{5} \cdot \left(-\sin \frac{t}{5}\right) = -2 \sin \frac{t}{5} ,$$

and

$$f'' : t \rightarrow -\frac{2}{5} \cos \frac{t}{5} .$$

For $t = 5$ we have

$$f(5) = 10 \cos 1 \approx 5.4$$

$$f'(5) = -2 \sin 1 \approx -1.68$$

$$f''(5) = -\frac{2}{5} \cos 1 \approx -0.22$$

since $$\cos 1 \approx 0.54$$

and $$\sin 1 \approx 0.84 .$$

When $t = 5$ seconds, Q is about 5.4 feet to the right of 0. It is moving to the left at 1.68 ft./sec. approximately. The velocity is decreasing at 0.22 ft./sec. each second so that Q is speeding up at this rate.

Let P move around the circle of radius r as in Figure 4-4b but this time let us study the motion of the projection R of P on the y-axis (Figure 4-4c). The point R is at

$$y = r \sin \theta = r \sin \omega t.$$

If g is the function

(8) $g : t \to r \sin \omega t,$

then

(9) $g' : t \to r\omega \cos \omega t$

and

(10) $g'' : t \to -r\omega^2 \sin \omega t.$

The point R starts at $t = 0$ at the point 0 and oscillates up and down on the y-axis between B and B'. It completes an oscillation 0 after the time $\frac{2\pi}{\omega}$. Note that

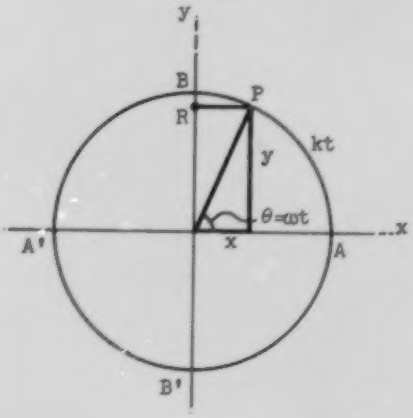


Figure 4-4c

$$g''(t) = -\omega^2 g(t)$$

or more briefly

$$g'' = -\omega^2 g.$$

We have discussed the motion of each of the points Q and R indicated in Figure 4-4c.

However, we may interpret $f'(t)$ and $g'(t)$ as the horizontal and vertical components of the motion of P itself. Notice that

$$f'(t) = -\omega g(t),$$
$$g'(t) = \omega f(t).$$

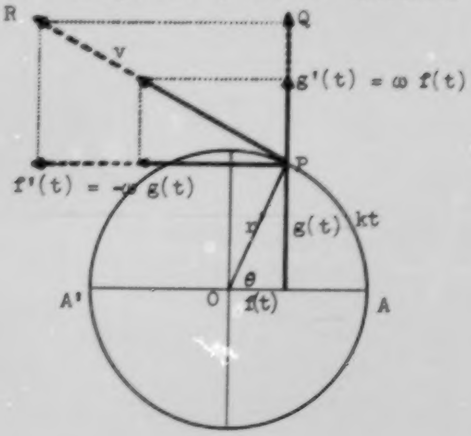


Figure 4-4d

The position and velocity of Q at time t are given by

$$f(t) = r \cos \omega t$$

$$f'(t) = -r\omega \sin \omega t;$$

and of R by

$$g(t) = r \sin \omega t$$

$$g'(t) = r\omega \cos \omega t.$$

The actual velocity of P is found by using the Pythagorean Theorem in triangle PQR:

$$\begin{aligned} v &= \sqrt{[f'(t)]^2 + [g'(t)]^2} \\ &= \sqrt{\omega^2[g(t)]^2 + \omega^2[f(t)]^2} \\ &= \omega\sqrt{[g(t)]^2 + [f(t)]^2} \\ &= \omega r. \end{aligned}$$

Since $\omega = \frac{k}{r}$, $\omega r = k$; so that, as we might expect, v is the constant rate k at which the arc is changing with time. You can easily convince yourself that the direction of v is perpendicular to the radius $\overline{OP}$, so that the motion of P is in the direction of the tangent, as it must be.

If we work with the horizontal and vertical components of acceleration of P in the same way, we find that

$$\begin{aligned} a &= \sqrt{[f''(t)]^2 + [g''(t)]^2} \\ &= \sqrt{[-\omega^2 f(t)]^2 + [-\omega^2 g(t)]^2} \\ &= \omega^2 \sqrt{[f(t)]^2 + [g(t)]^2} \\ &= \omega^2 r \end{aligned}$$

and that the direction of the acceleration is toward O.

Exercises 4-4

1. Suppose P moves around the circle at the rate of π arc units per second. Find the horizontal and vertical position, velocity, and acceleration when the following values are given the radius and time.

   (a) $r = 1$

   (i) $t = 0$
   (ii) $t = \frac{1}{2}$
   (iii) $t = 1$
   (iv) $t = 2$

   (b) $r = 2$

   (i) $t = 0$
   (ii) $t = 1$
   (iii) $t = 2$
   (iv) $t = 4$

   (c) $r = 6$

   (i) $t = 0$
   (ii) $t = 1$
   (iii) $t = 2$
   (iv) $t = 3$

2. Using the results of Number 1 compare the position, the horizontal and the vertical velocity, and the horizontal and the vertical acceleration of the following if

   (a) $r = 1$ and $t = \frac{1}{2}$;

   (b) $r = 2$ and $t = 1$; and

   (c) $r = 6$ and $t = 3$.

3. Using the results of Number 1 compare the horizontal and vertical position, the horizontal and the vertical velocity, and the vertical acceleration when $t = 0$, and again when point P has traversed the entire circle for

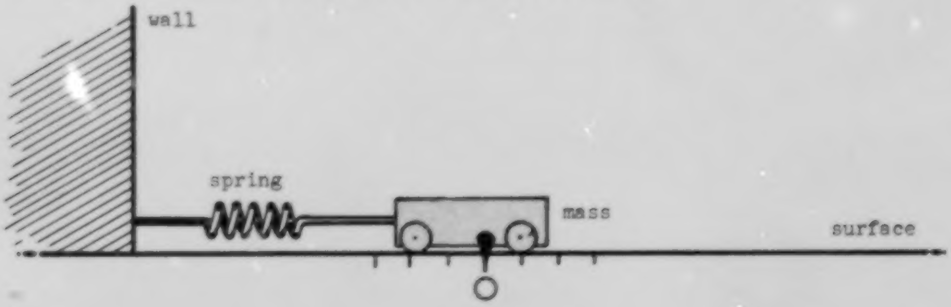
   (a) $r = 1$ (i.e., at $t = 0$ and at $t = 2$)

   (b) $r = 2$ (i.e., at $t = 0$ and at $t = 4$)

   (c) $r = 6$ (i.e., at $t = 0$ and at $t = 12$)

4. Show that the square root of the sum of the squares of the acceleration components is the product of the speed and the angular velocity; i.e., prove that $\sqrt{[f''(t)]^2 + [g''(t)]^2} = s \cdot \omega$, where s is the speed and ω the angular velocity.

5. Suppose a mass is attached to a spring as shown.



When the spring is unstretched, the pointer is at 0 on the scale. Suppose that we move the mass 3 units to the right and release it. The mass will oscillate back and forth. It can be shown that the spring acts in such a way that the position of the pointer is given by

$$f(t) = A \cos \sqrt{\frac{k}{m}}\, t$$

where A is a certain number and k and m are constants that measure the stiffness of the spring and the mass of the cart.

(a) What is the value of A?

(b) What is the period of the motion?

(c) Show that $f'(0) = 0$; that is, the initial velocity of the spring is zero.

(d) Show that $f''(0) = -A\frac{k}{m}$. Interpret the negative sign.

(e) Show that f is a solution to the differential equation $f'' = -\frac{k}{m} f$.

(f) How far left does the pointer move?

(g) Show that at time $t = \frac{\pi}{2}\sqrt{\frac{m}{k}}$ the pointer crosses the point 0. What are its velocity and acceleration at this time? In what direction is the mass moving at this time?

(h) At what time does the pointer cross 0 again? Find its velocity and acceleration at this time. In what direction is the mass moving at this time?

6. Draw a figure to justify the statement that in the case of uniform motion in a circle, the acceleration is toward the center. (Draw arrows to show the horizontal and vertical acceleration of P.)

4-5. Higher Derivatives and Approximations

When we differentiated a polynomial function (Chapter 2) we got another polynomial function. We discovered that if we took the first, second and higher derivatives of a polynomial function, we eventually got a derivative with value zero for all values of x. This will not be the case for a circular function, since the derivative of a circular function is a circular function. We shall see that if we take the derivative of a circular function, and then the derivative of the derivative, etc., we will soon obtain the very function with which we began.

For example, if we begin with the function

$$f : x \to \sin x$$

we get:

the first derivative $x \to \cos x$
the second derivative $x \to -\sin x$
the third derivative $x \to -\cos x$
and the fourth derivative $x \to \sin x$.

As was the case for polynomial functions we can write $Df = f'$, $D^2f = f''$, $D^3f = f'''$, $D^4f = f^{iv}$, $D^5f = f^{v}$, etc. The Roman superscript notation becomes cumbersome for high orders and it becomes more convenient to use Hindu-Arabic numerals parenthetically as $D^{13}f = f^{(13)}$. Thus the n-th derivative[*] of f is written $D^nf = f^{(n)}$. (It is also a useful convention to define the zero-order derivative of f as f itself: $f^{(0)} = f$.)

If we take successive derivatives of the sine function we get

$$(1)\qquad \begin{aligned} f' &: x \to \cos x \\ f'' &: x \to -\sin x \\ f''' &: x \to -\cos x \\ f^{(4)} &: x \to \sin x. \end{aligned}$$

Since the fourth derivative is the original function we can see the pattern:

---

[*] In Leibnizian notation we write

$$D^ny = \left(\frac{d}{dx}\right)^n y = \frac{d^ny}{dx^n}.$$

This can be further abbreivated: we write $y^{(n)}$, when we mean D^ny.

$$\text{(1a)} \quad \begin{aligned} f &= f^{(4)} = f^{(8)} = \dots \\ f' &= f^{(5)} = f^{(9)} = \dots \\ f'' &= f^{(6)} = f^{(10)} = \dots \\ f''' &= f^{(7)} = f^{(11)} = \dots \text{ etc.} \end{aligned}$$

This result is sometimes summarized by saying that each function $x \to y$, where $y = \sin x, \cos x, -\sin x$ or $-\cos x$ is a solution of the differential equation

$$\text{(2)} \qquad y^{(4)} = y,$$

where $y^{(4)}$ represents the value of the fourth derivative of $x \to y$.

We know from our polynomial discussion that the process of differentiation lowers the degree of the polynomial function. Thus, if $f : x \to y$ is a polynomial function of degree n then $f(x)$ is a solution to the equation

$$\text{(3)} \qquad y^{(n+1)} = 0.$$

The first derivative can be interpreted as the slope of a tangent line or as velocity; the second derivative can be thought of as the rate of change of the slope function or as acceleration. While physical and geometrical interpretations of higher derivatives are more difficult to contemplate, higher derivatives are useful in approximation discussions.

Approximations

We want to find a polynomial function which approximates the sine function. To do this we turn to the problem of finding a polynomial function whose first and higher derivatives "fit" the sine function near zero. More precisely for each positive integer n we shall show that there is a unique polynomial function p which satisfies the conditions:

(4)
- (a) the degree of $p \leq n$
- (b) $p(0) = 0 = \sin 0$
- (c) the values of the first n derivatives of p and the sine function are the same for $x = 0$.

To construct such a polynomial function p we need to find the coefficients: $a_0, a_1, a_2, \dots, a_n$ for

$$\text{(5)} \qquad p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n.$$

Since conditions (4b) and (4c) determine the values

$$p(0),\ p'(0),\ p''(0),\ \ldots,\ p^{(n)}(0)$$

we need only show that these values determine the coefficients of $p(x)$ in (5).

First try to approximate the sine function by a first degree polynomial function; i.e., we suppose that $n = 1$ so that

$$p(x) = a_0 + a_1 x$$

and

$$p'(x) = a_1.$$

Therefore, we have

(6) $$p(0) = a_0 \text{ and } p'(0) = a_1.$$

For $f : x \to \sin x$,

$$f(x) = \sin x \text{ and } f'(x) = \cos x$$

we have

(7) $$f(0) = 0,\ f'(0) = 1.$$

If the polynomial function p is to satisfy the conditions of (4) we require that

$$p(0) = f(0) \text{ and } p'(0) = f'(0).$$

Combining these requirements with (6) and (7), we conclude that

$$a_0 = 0 \text{ and } a_1 = 1.$$

Thus there is exactly one polynomial function p of degree $n \le 1$ such that $p(0) = f(0)$ and $p'(0) = f'(0)$; and that function is given by $p(x) = x$.

Consider now the case when $n = 2$ so that p has the form

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

In this case, we have

$$p'(x) = a_1 + 2a_2 x \text{ and } p''(x) = 2a_2$$

so that

(8) $$p(0) = a_0,\ p'(0) = a_1,\ p''(0) = 2a_2.$$

Since $f : x \to \sin x$, we have (from (1))

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x$$

so that

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0. \tag{9}$$

If p is to satisfy the conditions of (4) with $n = 2$ we require that

$$p(0) = f(0), \quad p'(0) = f'(0), \quad p''(0) = f''(0). \tag{10}$$

Combining the requirements of (10) with (8) and (9), we conclude that

$$a_0 = 0, \quad a_1 = 1, \quad \text{and} \quad 2a_2 = 0.$$

The cases $n = 1$ and $n = 2$ result in the same polynomial p given by $p(x) = x$. This result is not surprising since the second derivative of the sine function is zero when $x = 0$. Now we turn to the case for which $n = 3$; i.e., we suppose that p is a polynomial function of degree 3 given by

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$$

Differentiating, we get

$$p'(x) = a_1 + 2a_2x + 3a_3x^2,$$

$$p''(x) = 2a_2 + 6a_3x,$$

and

$$p'''(x) = 6a_3,$$

For $x = 0$ we have

$$p(0) = a_0, \quad p'(0) = a_1, \quad p''(0) = 2a_2, \quad \text{and} \quad p'''(0) = 6a_3. \tag{11}$$

Using (1) for $x = 0$ we obtain

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad \text{and} \quad f'''(0) = -1.$$

Comparing these values with (11) under the requirements of (4), we get

$$a_0 = 0, \quad a_1 = 1, \quad 2a_2 = 0, \quad \text{and} \quad 6a_3 = -1.$$

Therefore, the only polynomial p of degree 3 or less which satisfies the conditions of (4) is given by

$$p(x) = x - \frac{x^3}{6};$$

As we attempt to express a result for arbitrary n we shall use factorial notation

$$k! = 1 \cdot 2 \cdot 3 \ldots k \quad \text{and} \quad 0! = 1.$$

We can show that if

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n,$$

then we have

$$\begin{aligned} p(0) &= a_0 = (0!)a_0 \\ p'(0) &= a_1 = (1!)a_1 \\ p''(0) &= 2a_2 = (2!)a_2 \\ p'''(0) &= (2\cdot 3)a_3 = (3!)a_3 \\ p^{(4)}(0) &= (2\cdot 3\cdot 4)a_4 = (4!)a_4 \\ &\vdots \\ p^{(n)}(0) &= (2\cdot 3\cdot 4 \dots n)a_n = (n!)a_n. \end{aligned}$$

If $f : x \to \sin x$, then, from (1a), we can write

$$\begin{aligned} 0 &= f(0) = f^{(4)}(0) = f^{(8)}(0) = \dots \\ 1 &= f'(0) = f^{(5)}(0) = f^{(9)}(0) = \dots \\ 0 &= f''(0) = f^{(6)}(0) = f^{(10)}(0) = \dots \\ -1 &= f'''(0) = f^{(7)}(0) = f^{(11)}(0) = \dots \end{aligned}$$

To satisfy (4) we must have

$$p(0) = f(0), \quad p'(0) = f'(0), \quad p''(0) = f''(0), \dots .$$

Now we observe that

(13) all the coefficients of terms in p of even degree are 0.

The odd degree coefficients are

(14) $$a_1 = 1, \quad a_3 = -\frac{1}{3!}, \quad a_5 = \frac{1}{5!}, \quad a_7 = -\frac{1}{7!}, \dots .$$

For example, the polynomial p of degree 10 or less which satisfies the conditions (4) is given by

$$\begin{aligned} p(x) &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362,880}. \end{aligned}$$

In summary, for each positive integer n there is precisely one polynomial function p of degree not exceeding n, such that p and its first n

derivatives agree at $x = 0$ with the sine function and its first n derivatives. The polynomials p are known as the Taylor approximations at $x = 0$ to the sine function.

A similar process will yield the Taylor approximations to the cosine function:

$$1, \quad 1 - \frac{x^2}{2!}, \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!}, \ \ldots$$

In Exercises 4-5, Number 8, you are asked to obtain these approximations.

Approximation Error.

Now we want to determine the accuracy of successive Taylor approximations. The approximations

$$y = x, \quad y = x - \frac{x^3}{3!}, \quad y = x - \frac{x^3}{3!} + \frac{x^5}{5!}, \quad y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!},$$

and

$$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

to $y = \sin x$, are graphed in Figure 4-5a. Note that as the degree increases, the Taylor approximations become better in the sense that subsequent approximations improve the "fit" near zero and also give better approximations further away from zero.

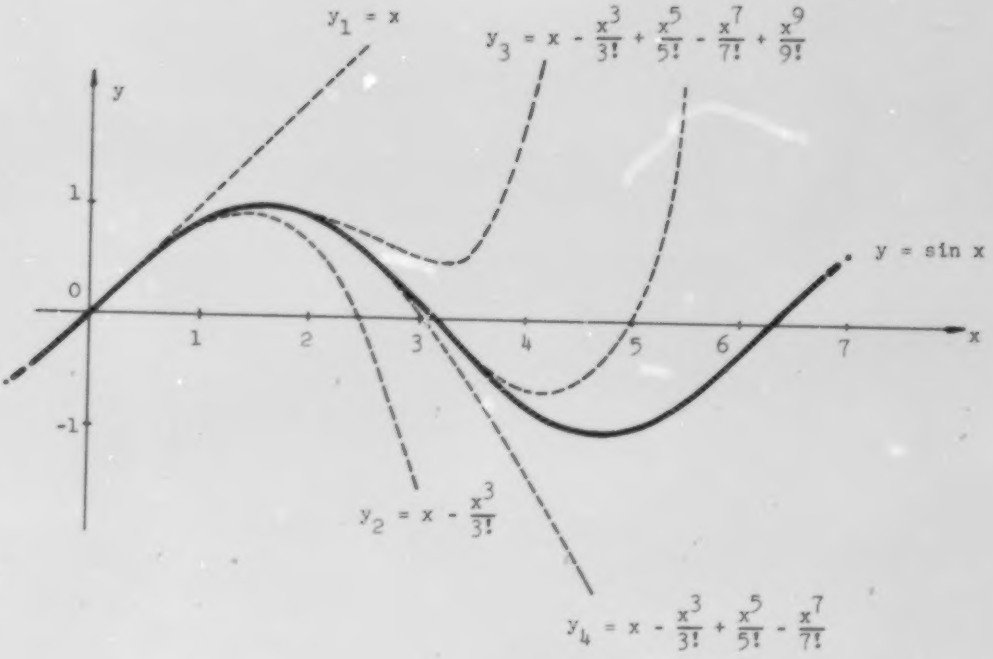


Figure 4-5a

We shall prove in Chapter 7 that these approximations are alternately too large and too small. In fact, for $x > 0$,

$$x > \sin x,$$

$$x - \frac{x^3}{3!} < \sin x,$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} > \sin x,$$

$$\cdots$$

Hence, the error made in using any one of these approximations is easy to estimate. For example, if we use the approximation

$$\sin x \approx x - \frac{x^3}{3!}$$

the result is too small by an amount which is less than $\frac{x^5}{5!}$. In practice x is less than 1, since there is no need to compute values of $\sin x$ for $x > \frac{\pi}{4}$. In this case, each of the successive terms

$$\frac{x^3}{3!},\ \frac{x^5}{5!},\ \frac{x^7}{7!},\ \ldots$$

is smaller than the preceding one. Moreover, since $\frac{x^n}{n!} < \frac{1}{n!}$ when $x < 1$, we can approximate $\sin x$ as closely as we please by choosing n large enough.

Similarly, it will be shown that the approximations

$$1,$$

$$1 - \frac{x^2}{2},$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

to $\cos x$ are alternately too large and too small so that, for $x > 0$,

$$1 > \cos x,$$

$$1 - \frac{x^2}{2!} < \cos x,$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} > \cos x,$$

$$\ldots$$

Hence, for example, the error in using the approximation $1 - \frac{x^2}{2!}$ is between 0 and $\frac{x^4}{4!}$.

Example 4-5a. Use Taylor approximations to estimate $\sin 0.5$.

Let us begin with the approximation

$$\sin x \approx x$$

which gives $\sin 0.5 \approx 0.5$. The result is too large by an amount less than $\frac{x^3}{3!} = \frac{0.125}{6} = 0.020833\ldots$

If we use the approximation

$$\sin x \approx x - \frac{x^3}{3!}$$

we obtain $\sin 0.5 \approx 0.5 - 0.020833\ldots = 0.479166\ldots$ This estimate is too small by an amount less than $\frac{x^5}{5!} = \frac{0.03125}{120} \approx 0.00026$. We can therefore conclude that to 3 decimal place accuracy $\sin 0.5 \approx 0.479$.

Example 4-5b. Use the approximation $1 - \frac{x^2}{2!}$ for $\cos x$ to estimate $\cos 0.2$. Also estimate the error committed in using this approximation.

$$\cos 0.2 \approx 1 - \frac{(0.2)^2}{2!} = 1 - \frac{0.04}{2} = 0.98.$$

The error is less than $\frac{x^4}{4!} = \frac{0.0016}{24} \approx 0.000067$.

It is often useful to write our results in terms of a remainder R. For example,

$$\sin x = x - \frac{x^3}{3!} + R,$$

where $$0 < R < \frac{x^5}{5!},$$

and $$\cos x = 1 - \frac{x^2}{2!} + R',$$

where $$0 < R' < \frac{x^4}{4!}.$$

We can use the Taylor approximations to determine certain limits. Let us begin with some familiar ones.

Example 4-5c. To find $\lim\limits_{x \to 0} \frac{\sin x}{x}$ and $\lim\limits_{x \to 0} \frac{1 - \cos x}{x}$.

We can write

$$\sin x = x - R, \text{ where } 0 < R < \frac{x^3}{3!}.$$

Hence,

$$\frac{\sin x}{x} = 1 - \frac{R}{x}.$$

Since $0 < \frac{R}{x} < \frac{x^2}{2!}$, we conclude that $\frac{R}{x}$ approaches zero and $\frac{\sin x}{x}$ approaches 1.

Similarly,

$$\cos x = 1 - \frac{x^2}{2!} + R', \quad (0 < R' < \frac{x^4}{4!})$$

Hence, $$1 - \cos x = \frac{x^2}{2!} - R'$$

and
$$\frac{1 - \cos x}{x} = \frac{x}{2!} - \frac{R'}{x}.$$

Since $\frac{x}{2!}$ and $\frac{R'}{x}$ both approach zero
$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$

Example 4-5d. To find $\lim_{x \to 0} \frac{1 - \cos x}{\sin x}$.

Since
$$\cos x = 1 - \frac{x^2}{2!} + R', \quad (0 < R' < \frac{x^4}{4!})$$

and
$$\sin x = x - R, \qquad (0 < R < \frac{x^3}{3!})$$

we have
$$\frac{1 - \cos x}{\sin x} = \frac{\frac{x^2}{2!} - R'}{x - R} = \frac{\frac{x}{2!} - \frac{R'}{x}}{1 - \frac{R}{x}}.$$

The numerator approaches 0 and the denominator approaches 1. Hence, the required limit is zero.

This process can be abbreviated by recognizing that
$$\cos x \approx 1 - \frac{x^2}{2}$$

and
$$\sin x \approx x.$$

Therefore,
$$\frac{1 - \cos x}{\sin x} \approx \frac{1 - (1 - \frac{x^2}{2})}{x} = \frac{\frac{x^2}{2}}{x} = \frac{x}{2},$$

which approaches 0 as x approaches 0.

This example may be done in still another way. We may write

$$\frac{1 - \cos x}{\sin x} = \frac{1 - \cos x}{x} \cdot \frac{x}{\sin x} .$$

From the previous example $\frac{1 - \cos x}{x}$ approaches 0 and $\frac{x}{\sin x}$ approaches 1. We can multiply these two limits to obtain the required limit.

It can be shown that the Taylor approximations are the best polynomial approximations near zero. For example, if $f : x \to \sin x$ and $p : x \to x - \frac{x^3}{3!}$ then

$$\frac{f(x) - p(x)}{x^3} \text{ approaches } 0$$

as x approaches 0; and p is the only polynomial of degree 3 or less with this property.

Exercises 4-5

1. (a) Given: $f : x \to \sin x$, determine f', f'', f''', $f^{(4)}$, then find

(i) $f^{(21)}(x)$ (iii) $f^{(35)}(x)$

(ii) $f^{(18)}(x)$ (iv) $f^{(16)}(x)$

(b) Given: $g : x \to \cos x$, determine g', g'', g''', $g^{(4)}$, then find

(i) $g^{(31)}(x)$ (iii) $g^{(20)}(x)$

(ii) $g^{(42)}(x)$ (iv) $g^{(101)}(x)$

2. (a) Given: $f : x \to \sin x$, determine the value of

(i) $f'(\frac{\pi}{6})$ (iii) $f'''(\frac{\pi}{4})$

(ii) $f''(\frac{\pi}{3})$ (iv) $f^{(4)}(\frac{5\pi}{6})$

(b) Given: $g : x \to \cos x$, find

(i) $g'(\frac{\pi}{6})$ (iii) $g'''(\frac{2\pi}{3})$

(ii) $g''(\frac{3\pi}{4})$ (iv) $g^{(4)}(\frac{5\pi}{6})$

3. (a) Given: $f : x \to A \sin ax$, find

(i) $f'(x)$ (iii) $f'''(x)$

(ii) $f''(x)$ (iv) $f^{(4)}(x)$

(b) Given: $g : x \to B \cos bx$, find

(i) $g'(x)$ (iii) $g'''(x)$

(ii) $g''(x)$ (iv) $g^{(4)}(x)$

4. (a) Given: $f : x \to 3 \sin \pi x$, find

(i) $f'(x)$ (iii) $f'''(x)$

(ii) $f''(x)$ (iv) $f^{iv}(x)$

(b) Given: $g : x \to 2 \cos \frac{x}{2}$,

(i) $g'(x)$ (iii) $g'''(x)$

(ii) $g''(x)$ (iv) $g^{iv}(x)$

5. If $f : x \to 3 \sin (\frac{x}{2} + \frac{\pi}{4})$, find

   (a) $f(0)$

   (b) (i) $f'(x)$ (ii) $f'(\pi)$

   (c) (i) $f''(x)$ (ii) $f''(\frac{\pi}{2})$

   (d) (i) $f'''(x)$ (ii) $f'''(-\frac{3\pi}{2})$

   (e) (i) $f^{(4)}(x)$ (ii) $f^{(4)}(2\pi)$

6. If $g : x \to -2 \cos (2x + \frac{\pi}{2})$, find

   (a) $g(0)$

   (b) (i) $g'(x)$ (ii) $g'(\pi)$

   (c) (i) $g''(x)$ (ii) $g''(-\frac{\pi}{12})$

   (d) (i) $g'''(x)$ (ii) $g'''(0)$

   (e) (i) $g^{(4)}(x)$ (ii) $g^{(4)}(\frac{\pi}{2})$

7. Find a formula for the nth derivative of the sine function.

8. Show that the Taylor approximations to $x \to \cos x$ are given by

$$1, \quad 1 - \frac{x^2}{2!}, \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!}, \quad \dots$$

   and, in general, by

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^k \frac{x^n}{n!}$$

   where n is even and $k = \frac{n}{2}$.

For Numbers 9, 10, and 12, use the Taylor approximation for sine and cosine.

9. (a) Calculate $\sin(0.2)$ using $n = 4$.

   (b) Estimate the error.

10. (a) Calculate $\cos(0.2)$ using $n = 4$.

   (b) Estimate the error.

11. (a) Calculate $\cos \frac{1}{2}$ using

    (i) $n = 2$
    (ii) $n = 4$
    (iii) $n = 6$

    (b) In each case estimate the error, ascertaining the number of places of accuracy in the approximation of $\cos \frac{1}{2}$.

12. Show that the sine function is not equal to a polynomial function on any interval; i.e., given $a < b$ and any polynomial p, there is a number x such that

$$a \leq x \leq b \quad \text{and} \quad \sin x \neq p(x).$$

(Hint: Suppose $\sin x = p(x)$, $a \leq x \leq b$ and differentiate several times.)

13. Find the limits of the following expressions as x approaches 0.

(a) $\dfrac{(\sin x) - (x - \frac{x^3}{6})}{x^5}$

(b) $\dfrac{\sin x^2 - (x^2 - \frac{x^6}{6})}{x^{10}}$ (Let $x^2 = t$ and find the limit as $t \to 0$)

(c) $\dfrac{4x^2 - \sin 4x^2}{x^6}$ (Let $4x^2 = t$)

(d) $\dfrac{3x - \sin 3x}{6x}$ (Let $3x = t$)

(e) $\dfrac{1 - \cos x^2}{x^4}$

(f) $\dfrac{1 - \cos x^3}{x^4}$

14. Use Taylor approximations for $\sin x$ and $\cos x$ to evaluate each of the following limits (if the limit exists).

(a) $\lim\limits_{x \to 0} \dfrac{\sin x}{\cos x}$

(b) $\lim\limits_{x \to 0} \dfrac{x \cos x - \sin x}{x}$

(c) $\lim\limits_{x \to 0^+} \dfrac{\sin x}{\sqrt{x}}$

(Note: "$x \to 0^+$" means that x approaches zero from the right only.)

15. Find the limit of each of the following expressions as x approaches 0.

(a) $\dfrac{\sin x}{1 + \cos x}$

(b) $\dfrac{\sin^2 x}{1 - \cos x}$

(c) $\dfrac{x^2 \sin x^2}{(1 - \cos x)^2}$

16. Show that the Taylor approximations to $x \to \sin 2x$ can be obtained by replacing x by $2x$ in the approximations for $x \to \sin x$.

17. Taylor approximations at $a \neq 0$

(a) Show that for each positive integer n, there is a unique polynomial p such that

(i) $\deg p \leq n$

(ii) $p(a) = \sin a$

(iii) the first n derivatives of p have the same respective values at $x = a$ as the first n derivatives of $f : x \to \sin x$.

These polynomials are called the Taylor approximations of the sine function at $x = a$.

(b) Show that for $n = 2$

$$p(x) = \sin a + (\cos a)(x - a) - \frac{\sin a}{2!}(x - a)^2.$$

(c) Show that, in general,

$$p(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

(d) Derive a formula corresponding to that in part (b) for the cosine Taylor approximations.

Chapter 5

EXPONENTIAL AND RELATED FUNCTIONS

In the preceding four chapters we studied polynomial and circular functions, the former being defined algebraically, the latter in terms of arc length on the unit circle. In this and the next chapter we take up the study of exponential functions and several other functions which can be described in terms of exponentials. In order to define an exponential function, such as $x \to 2^x$, we must show how irrational powers are defined. Then we must obtain the properties of these functions.

In order to provide a basis for our development of exponential functions and their inverses, the logarithm functions, we begin this chapter with a review of the laws of exponents. To assist us in our discussion of exponential functions, we show in Section 5-2 how the function $x \to N2^x$ serves as a model for growth. The laws of rational exponents and the fact that 2^x increases as x (rational) increases are established in Section 5-3. A method for defining irrational powers of 2 is indicated in Section 5-4, where it is noted that laws of exponents for arbitrary real numbers hold and that the graph of the resulting function $x \to 2^x$ is rising and has no gaps. These facts are used in Section 5-5 to obtain the definitions and properties of the general exponential function. The final two sections use the inverse concept to define and analyze the logarithm functions.

5-1. Exponents

You are doubtless familiar with the behavior of exponents. Since, however, we shall need to use them extensively, it seems wise to put our knowledge of them in order.

Let us consider the sequence of numbers

$$2,\ 2 \times 2,\ 2 \times 2 \times 2,\ 2 \times 2 \times 2 \times 2,\ \ldots$$

which we abbreviate as

$$2^1,\ 2^2,\ 2^3,\ 2^4,\ \ldots \tag{1}$$

using the exponents $1, 2, 3, 4$ to indicate the number of equal factors.

Can we give meaning to

$$2^0,\ 2^{-1},\ 2^{-2},\ \ldots$$

or to

$$2^{1/2},\ 2^{2/3},\ \ldots?$$

Certainly we cannot talk sensibly about -2 equal factors or $\frac{1}{2}$ equal factors.

If we look at the list (1) we note that addition of 1 to the exponent results in a multiplication by 2. Thus $2^4 = 2^{3+1} = 2 \cdot 2^3$. We can restate this principle by saying that subtraction of 1 from the exponent gives a result which is $\frac{1}{2}$ the original one. Thus,

$$2^3 = \frac{1}{2}(2^4),\ 2^2 = \frac{1}{2}(2^3),\ \ldots$$

What number shall 2^0 represent? We can get to 2^0, by subtracting 1 from the exponent 1 in 2^1. Therefore, if we are to maintain the principle, we should have

$$2^0 = \frac{1}{2}(2^1) = 1.$$

Subtracting another 1 from the exponent, we get

$$2^{-1} = \frac{1}{2}(2^0) = \frac{1}{2} \cdot 1 = \frac{1}{2}\ .$$

Continuing in this fashion we find successively

$$2^{-2} = \frac{1}{4},$$
$$2^{-3} = \frac{1}{8},$$
$$2^{-4} = \frac{1}{16},$$

and generally

$$2^{-n} = \frac{1}{2^n}$$

for every positive integer n.

To interpret $2^{1/2}$ we shall assume that equal increases in the exponent correspond to equal ratios of the numbers. With integer exponents this principle takes the form that each increase of 1 in the exponent corresponds to a ratio of 2. What ratio corresponds to an increase of $\frac{1}{2}$? Let us call this unknown ratio r. Then

$$2^{1/2} = r \cdot 2^0 = r \cdot 1 = r$$

and
$$2^{1/2+1/2} = r \cdot 2^{1/2} = r^2.$$

But
$$2^{1/2+1/2} = 2^1 = 2.$$

Hence,
$$r^2 = 2$$

and
$$r = \sqrt{2}.$$

We reject the possibility $r = -\sqrt{2}$ which would not fit nicely on the graph. (See Figure 5-1a.)

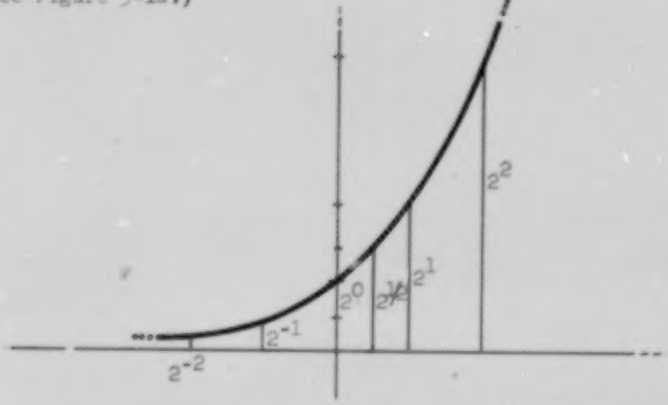


Figure 5-1a

Powers of 2

A similar argument shows that to maintain regularity we should take

$$2^{1/3} = \sqrt[3]{2}$$

$$2^{2/3} = \sqrt[3]{2^2}.$$

and so on. Generalizing, we are led to define

$$2^{p/q} = \sqrt[q]{2^p}$$

where p and q are positive integers.

We can reverse our principle and say that every time we subtract a given amount from the exponent we divide the number by a fixed amount. Then we conclude that

$$2^{-1/2} = \frac{1}{\sqrt{2}} 2^0 = \frac{1}{\sqrt{2}}, \quad 2^{-2/2} = \frac{1}{2}, \quad 2^{-3/2} = \frac{1}{2\sqrt{2}} = \frac{1}{2^{3/2}}$$

and so on.

With these interpretations of negative integral exponents and functional exponents, we see that if r and s are any two rational numbers then

$$2^r \cdot 2^s = 2^{r+s}$$

and

$$(2^r)^s = 2^{rs}.$$

More generally, if a is any positive number and if r and s are two rational numbers then it turns out that

(2)
$$a^r a^s = a^{r+s}$$

and

(3)
$$(a^r)^s = a^{rs}.$$

These equations express the familiar laws of exponents.

Exercises 5-1

1. Write each of the following as a positive power of one number

   (a) $x^5 \cdot x^{-2}$

   (b) $10^9 \cdot 10^{-7}$

   (c) $a^{3/5} \cdot a^{5/3}$

   (d) $2^2(\frac{2^4}{2^8})$

   (e) $2^{3/4} \cdot 8^{4/3}$

   (f) $(a^{3/5})^{5/3}$

   (g) $\frac{64^{2/3}}{32^{3/10}}$

   (h) $\frac{3 \cdot 2^{1/2} - 8^{1/2}}{32^{1/2}}$

   (i) $\frac{3^2 \cdot 2^2}{(6^{2/3})^0}$

   (j) $\frac{5^{1.1}}{25^{.55}}(\frac{1}{125})^{-2/3}$

2. Find the value of m if:

   (a) $8^m = (2^3)^2$;

   (b) $8^m = 2^{(3^2)}$;

   (c) $2^{(4^5)} = 16^m$;

   (d) $(2^4)^5 = 16^m$;

   (e) $4^{m-1} = 64^{2/3}$;

   (f) $5^{m-1} = 0.2$;

   (g) $(\frac{3}{2})^m = \frac{2}{3}$;

   (h) $17^m = 1$

3. Evaluate: $1000(8^{-2/3})$, $3(\frac{9}{4})^{-3/2}$.

4. Arrange the following in order of magnitude:

$$2^{2/3},\quad (4^{5/2})(8^{-1}),\quad (\tfrac{1}{2})^{-4/3},\quad 2^{-3},\quad (2^{-2/9})^9 .$$

5. Show that if $x = 2^{2.7}$, then $x = 4\sqrt[10]{128}$.

6. Carry out an argument like that in the text to show that

   (a) $3^{1/4} = \sqrt[4]{3}$

   (b) $4^{-3} = \frac{1}{4^3}$

   (c) $2^{-1/2} = \frac{1}{2^{1/2}}$

5-2. The Exponential Function, Growth and Decay

In this and the next three sections we shall be concerned with assigning a meaning to a^x where $a > 0$ and x is an arbitrary real number. This will lead us to the general exponential function

$$x \to ka^x$$

where k is a constant. The number a is called the base of the exponential function.

An exponential function can serve as an idealized mathematical model for growth and decay. Let us consider growth. Suppose a biologist grows a colony of a certain kind of bacteria. He wishes to study how the number of bacteria changes with time. Under favorable circumstances he finds that so long as the food holds out, the time required for the number of bacteria to double does not seem to depend on the time at which he starts the experiment. His hypothesis is that the time required for the bacteria to double does not depend upon the time when the initial count is made. This is one instance of a general growth principle which is important in social, physical, and biological science.

To be concrete, let us suppose that on a given day there are N_0 bacteria present and that the number of bacteria doubles every day. Then there will be $2N_0$ present one day later. After another day the number of bacteria will be twice $2N_0$ or 2^2N_0, after three days twice 2^2N_0 or 2^3N_0. After n days the number $N(n)$ of bacteria present will be given by the equation

$$N(n) = N_0 2^n, \tag{2}$$

where n is a positive integer.

If we assume that the number of bacteria increases steadily throughout any given day, we might want to determine how many bacteria are present $\frac{1}{2}$ day after the start or how many were present $2\frac{1}{2}$ days before the initial count was made.

To answer these questions we must generalize equation (2) to

$$N(r) = N_0 2^r,$$

where r may take any rational values, positive or negative. If such expressions as $2^{\sqrt{2}}$ and 2^π are to have meaning, we must generalize further to

$$N(x) = N_0 2^x, \tag{3}$$

where x is an arbitrary real number.

Let us suppose that (3) serves as a model for the growth of the bacteria colony. Can we deduce from this growth equation that the time required for the bacteria to double doesn't depend on the time the initial count is made? Since this was the initial hypothesis, it should be true if (3) is to serve as a model. Since we can be certain of the meaning of 2^x only when x is rational (from Section 5-1) we begin with that case to check the hypothesis. Suppose we take a count x days after the start of the experiment and then take another count t days later. Here x and t may be any rational numbers, positive, negative, or zero. From (3) we have

$$N(x + t) = N_0 2^{x+t}.$$

From Section 5-1, the exponent law ($2^{r+s} = 2^r 2^s$, for r and s rational) gives

$$N(x + t) = N_0 2^x \cdot 2^t.$$

Replacing $N_0 2^x$ by $N(x)$ we obtain

$$N(x + t) = 2^t N(x). \tag{4}$$

In words, if $N(x)$ is the bacterial count after x days, then the number $N(x + t)$ of bacteria after $x + t$ days is 2^t times as great. The factor 2^t does not depend upon x, the time of the first count; it depends only upon t, the time interval between counts.

For example, suppose that there are one million bacteria present initially; i.e., $N_0 = 10^6$. Then the number of bacteria one-half day later is given by

$$\begin{aligned} N(\tfrac{1}{2}) &= 10^6(2^{1/2}) \\ &= 10^6 \sqrt{2}\ . \end{aligned}$$

After one and one-half days the number of bacteria in the colony is:

$$\begin{aligned} N(\tfrac{3}{2}) &= 10^6(2^{3/2}) \\ &= 10^6(2\sqrt{2}) \\ &= 2(10^6\sqrt{2}) \\ &= 2\ N(\tfrac{1}{2}) \end{aligned}$$

If we use (4) we obtain the same result:

$$N(\tfrac{3}{2}) = N(\tfrac{1}{2} + 1) = 2N(\tfrac{1}{2}).$$

Assuming that the conditions of growth were the same prior to the initial count, we should expect that the number of bacteria one day before the initial count is taken would be

$$N(-1) = 10^6 \cdot 2^{-1} = 500{,}000.$$

An exponential function with any positive base can serve as a growth model. If we replace the assumption that the number of bacteria doubles each day by the assumption that the number changes by a factor of a each day, we obtain the model

$$N(x) = N_0 a^x.$$

Since the laws of exponents hold for rational exponents and an arbitrary base we obtain the relation

$$N(x + t) = a^t\, N(x).$$

We see that it is also true in the general case that the growth factor (a^t) depends only upon the length of the time interval (t) and not on the first observation time (x).

Exercises 5-2

In the first four exercises consider the equation

$$N(n) = 10^6(2^n),$$

where $N(n)$ represents the number of bacteria present at the end of n days.

1. Plot the points for which $n = 0, 1, 2, 3, 4$ and connect them with a smooth curve. (The unit chosen for the vertical axis may be one million.)

2. The bacteria count at the end of $n + 5$ days is how many times as great as the count $n + 2$ days after the beginning of the experiment?

3. One week after the initial count was made the number of bacteria present was how many times as great as the number present three days before the experiment began?

4. If there are k bacteria present after 100 days, after how many days were there $\frac{k}{4}$ present?

5. Suppose that in a new experiment there are 200,000 bacteria present at the end of three days and 1,600,000 present at the end of $4\frac{1}{2}$ days. Compute:

   (a) the number present at the end of 5 days;

   (b) the number present at the end of $1\frac{1}{2}$ days;

   (c) the number of days at the end of which there are 800,000 bacteria present.

   Hint: Assume that the number of bacteria present at the beginning of the experiment is N_0 and that at the end of 24 hours the count is $a \cdot N_0$.

6. The number of bacteria in a certain culture is observed to double every day. If there were 10^5 present at the first count, the number of bacteria present after t days is given by $N(t) = 10^5 \cdot 2^t$.

   (a) How many were there after 2 days? 4 days?

   (b) How many were there one day before the count? two days before the count?

   (c) How many were there one-half day before the count? one-half day after the count? What is the ratio of $N(\frac{1}{2})$ to $N(-\frac{1}{2})$?

7. Suppose $N(t) = N_0a^t$ is the number of bacteria present at time t.

   (a) If this formula is to represent growth, can a be smaller than 1? Use the formula with $a = \frac{1}{2}$, $a = 1$, and $a = \frac{3}{2}$ to sketch graphs illustrating your answer.

   (b) Calculate

   $$\frac{N(t+1)}{N(t)} ; \quad \frac{N(t+2)}{N(t+1)} ; \quad \frac{N(t+3)}{N(t+2)} .$$

   (c) In general what is the value of $\frac{N(t+n+1)}{N(t+n)}$?

   (d) Suppose that $N(1) = 10^5$ and $N(2) = 10^6$. Find N_0 and a.

8. A radioactive substance (such as radium) decays so that the amount present, $N(t)$, at time t is satisfactorily given by the same formula as growth:

   $$N(t) = N_0a^t, \text{ where } a \text{ is a positive real number.}$$

   (a) What is the amount present at time $t = 0$?

   (b) If this is to represent decay, can a exceed 1?

   (c) Show that the ratio of the amount present at time $t + 1$ to the amount present at time t doesn't depend upon t and is smaller than 1.

   (d) If $N(t) = N_0(\frac{1}{2})^t$, find the value of t for which $N(t) = \frac{N_0}{2}$.

5-3. More About Rational Exponents

Before attempting to define irrational exponents we shall further examine the function

$$r \to a^r$$

where $a > 0$ and r is a rational number. To be concrete let us suppose that $a = 2$; we consider the function

$$r \to 2^r, \quad r \text{ rational.} \tag{1}$$

For r and s rational, the laws of exponents are

$$\begin{aligned} 2^{r+s} &= 2^r 2^s, \\ (2^r)^s &= 2^{rs}. \end{aligned} \tag{2}$$

We can show that the function (1) is increasing; that is,

$$2^r < 2^s \text{ if } r \text{ and } s \text{ are rational and } r < s.$$

We first observe that

$$\text{if } a > 1, \text{ then } a^2 > 1, a^3 > 1, \ldots, a^n > 1, \text{ where } n \text{ is a positive integer.} \tag{3}$$

Similarly we note that

$$\text{if } a = 1, \text{ then } a^n = 1, \tag{4}$$

and

$$\text{if } 0 < a < 1, \text{ then } 0 < a^n < 1. \tag{5}$$

Now we assert that $2^{m/n} > 1$ for any positive integers m and n. If $a = 2^{m/n}$ were equal to 1, then (4) would lead to the result $a^n = 2^m = 1$, which is false. If, on the other hand, $2^{m/n}$ were less than 1, then (5) would lead to the result $a^n = 2^m < 1$, which is also false. Since $2^{m/n}$ is neither less than nor equal to 1, it must be greater than 1.

Now let r and s be any two rational numbers such that $r < s$. Then $s - r$ is a positive rational number $\frac{m}{n}$. In conclusion we have

$$\begin{gathered} 2^{s-r} = 2^{m/n} > 1 \\ 2^r(2^{s-r}) > 2^r \\ 2^s > 2^r. \end{gathered}$$

Therefore,

$$r \to 2^r, \text{ where } r \text{ is rational,}$$

is an increasing function.

If n is a large integer then certainly 2^n is very large. Estimates of the size of 2^n can be obtained by using the binomial theorem to expand $(1 + 1)^n$. For example, we have for $n > 1$

$$(1 + 1)^n = 1^n + n\, 1^{n-1} \cdot 1 + \frac{n(n - 1)}{2}\, 1^{n-2} \cdot 1^2 + \ldots$$

All of these terms are positive, so if we omit any terms we can only decrease the size. In particular, omitting all but the first and second terms we obtain

$$2^n = (1 + 1)^n > 1 + n. \tag{6}$$

Consequently, if we go far enough to the right the graph of

$$y = 2^x, \quad x \text{ rational} \tag{7}$$

must lie above any given horizontal line. Another consequence of (6) is the fact that the negative x-axis is an asymptote for the graph of (7); that is, the graph of (7) approaches the x-axis for x negative and $|x|$ large. To show this we can take reciprocals in (6) to obtain

$$\frac{1}{2^n} < \frac{1}{1 + n}\,. \tag{8}$$

Since $2^{-n} = \frac{1}{2^n}$ and since $\frac{1}{1 + n}$ approaches zero as $|n|$ increases, then

$$y = 2^x$$

approaches zero as x becomes negatively infinite.

In summary, the graph of $y = 2^x$, x rational, increases from near the x-axis (when $x < 0$ and $|x|$ is large), crossing the y-axis at $(0,1)$, and rises rapidly for $x > 0$.

Table 5-3. Values of 2^r

r	2^r	2^{-r}
.001	1.000 693 4	0.999 307 1
.005	1.003 471 7	0.996 540 2
.01	1.006 955 6	0.993 092 5
.02	1.013 96	0.986 23
.03	1.021 01	0.979 42
.04	1.028 11	0.972 66
.05	1.035 26	0.965 94
.10	1.071 77	0.933 03
.15	1.109 57	0.901 25
.20	1.148 70	0.870 55
.25	1.189 21	0.840 90
.30	1.231 14	0.812 25
.35	1.274 56	0.784 58
.40	1.319 51	0.757 86
.45	1.366 04	0.732 04
.50	1.414 21	0.707 11
.55	1.464 08	0.683 02
.60	1.515 72	0.659 75
.65	1.569 17	0.637 28
.70	1.624 50	0.615 57
.75	1.681 79	0.594 60
.80	1.741 10	0.574 35
.85	1.802 50	0.554 78
.90	1.866 07	0.535 89
.95	1.931 87	0.517 63
1.00	2.000 00	0.500 00

Rational Values of 2^r

Table 5-3 gives rational powers of 2. Ordinarily it is sufficient to use the entries to three place accuracy. The laws of exponents (2) can be used to find 2^r for values not listed in the table.

Example 5-3a. Find $2^{1.68}$.

We note that

$$\begin{aligned} 2^{1.68} &= 2^{(1 + 0.65 + .03)} \\ &= 2^1 \cdot 2^{0.65} \cdot 2^{0.03} \\ &\approx 2(1.569)(1.021) \approx 3.204 \end{aligned}$$

Example 5-3b. Find $2^{-0.37}$.

We write

$$\begin{aligned} 2^{-0.37} &= 2^{-1 + 0.63} \\ &= \frac{1}{2}(2^{0.60 + 0.03}) = \frac{1}{2}(2^{0.60})(2^{0.03}) \\ &\approx \frac{(1.516)(1.021)}{2} \approx 0.774 \end{aligned}$$

Example 5-3c. Find $4^{3.21}$.

Note that $4 = 2^2$, so that

$$\begin{aligned} 4^{3.21} &= (2^2)^{3.21} = 2^{2 \times 3.21} \\ &= 2^{6.42} = 2^6 \cdot 2^{0.40} \cdot 2^{0.02} \\ &= 64(1.320)(1.014) \approx 85.663. \end{aligned}$$

Later we shall be able to use Table 5-3 to calculate 8^r, 16^r and such expressions as 3^r. (In Section 5-5 we shall show how to define a^x, for general positive a, in terms of powers of 2.)

Exercises 5-3

1. Calculate $2^{5/4}$

   (a) by using the data in Table 5-3;

   (b) by noting that $2^{5/4} = 2 \cdot 2^{1/4} = 2\sqrt{\sqrt{2}}$.

2. Using the data in Table 5-3, calculate

   (a) $2^{1.15}$

   (b) $2^{2.65}$

   (c) $2^{0.58}$

   (d) $2^{-0.72}$

3. With the aid of Table 5-3, compute

   (a) $8^{0.84}$

   (b) $0.25^{-0.63}$

4. Extend Table 5-3 by completing the following table.

Table 5-3 (extended) Values of 2^r

r	2^r
-4.0	
-3.6	
-3.2	
-2.8	
-2.4	
-2.0	
-1.6	
-1.2	
1.4	
1.8	
2.2	
2.6	
3.0	

5. Plot the points $(x, 2^x)$ for the rational values of x shown in Table 5-3 and Table 5-3 extended (Number 4).

6. (a) For what positive values of the constant a is the function

$$f : r \rightarrow a^r$$

increasing? decreasing? constant?

(b) For what positive values of a is the function

$$f : r \to (a)^{r^2}$$

increasing? decreasing? constant?

(c) For what positive values of a is the function,

$$f : r \to (a)^{-r^2}$$

increasing? decreasing? constant?

(d) If $2b + 3 > 0$ for what values of b is the function

$$f : r \to (2b + 3)^r$$

increasing? decreasing? constant?

7. (a) Show that if $n \geq 2$ then

$$2^n > \frac{n(n - 1)}{2} ,$$

(b) Use (a) to show that

(i) $\frac{2^{100}}{100} > \frac{99}{2}$

(ii) $\frac{2^{10,000}}{10,000} > \frac{9,999}{2}$

(c) As n becomes large does $\frac{2^n}{n}$ become large? Justify your answer.

5-4. Arbitrary Real Exponents

In the preceding sections, we dealt with the properties of 2^x and a^x for x rational. We want to give meaning to these expressions if x is irrational. For example, we want to assign meaning to

$$2^{\sqrt{2}}, \ \pi^{\sqrt{3}}, \ 3^{-\pi}.$$

To be specific, we want the expression a^x to be defined in a natural way for irrational values of x; that is, we need to extend the function $x \to a^x$ so that its domain is the set of all real numbers x. To be concrete let us again suppose $a = 2$. In the next section we shall show how to define a^x, for general positive a, in terms of powers of 2.

Of course, the meaning of "in a natural way" is ambiguous. Geometrically, however, what we wish to do is clear. After plotting the points

$$(x\ , 2^x)$$

for a large number of rational values x, then we just connect these points with a smooth curve and obtain the desired graph of

$$x \to 2^x.$$

Then, for example, $2^{\sqrt{2}}$ is calculated by measuring the second coordinate of the point on this graph whose first coordinate is $\sqrt{2}$. (See Figure 5-4a.)

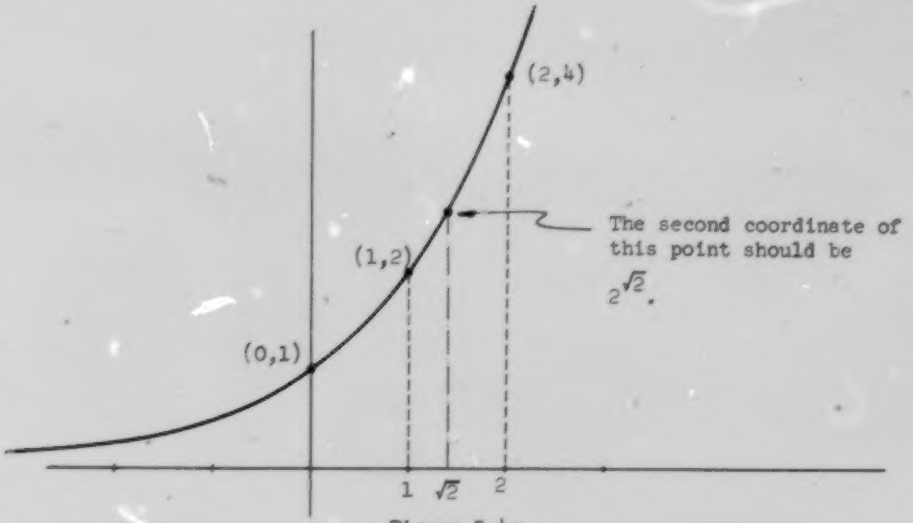


Figure 5-4a

Finding $2^{\sqrt{2}}$ by filling in the "holes" in $x \to 2^x$, x rational.

This graphic process is not quite satisfactory as it doesn't lead to precision as to the meaning of $2^{\sqrt{2}}$. We shall outline an approximation procedure which will enable us to define $2^{\sqrt{2}}$ and in general 2^x for irrational x.

The function so obtained is increasing everywhere; that is

(1) $$\text{if } u < v, \text{ then } 2^u < 2^v.$$

In fact, there is only one increasing function which has the values 2^r for all rational numbers r. For this function the laws of exponents

(2) $$2^{u+r} = 2^u 2^v \text{ and } (2^u)^v = 2^{uv}$$

hold for all real numbers u and v (rational or irrational). The graph of this function $x \to 2^x$ has no "gaps"; that is

(3) for any positive number y, there is an x such that $2^x = y$.

In the sequel we shall assume that indeed 2^x is so defined that (1), (2) and (3) are true and examine the consequences of these assumptions.

Now we turn to outlining the process used to define $2^{\sqrt{2}}$. If we wish the increasing property (1) to hold, then for all rational numbers r and s, such that

(4) $$r < \sqrt{2} < s$$

we must have

(5) $$2^r < 2^{\sqrt{2}} < 2^s.$$

Obviously this places a severe restriction on the value we assign to $2^{\sqrt{2}}$ and, as we shall see, determines it completely. The ordinary decimal approximations to $\sqrt{2}$ give us a handy collection of values for r and s; we know that

$$\begin{aligned}
1.4 &< \sqrt{2} < 1.5 \\
1.41 &< \sqrt{2} < 1.42 \\
1.414 &< \sqrt{2} < 1.415 \\
1.4142 &< \sqrt{2} < 1.4143 \\
1.41421 &< \sqrt{2} < 1.41422
\end{aligned}$$

and so on. The inequalities (4) and (5) then show that $2^{\sqrt{2}}$ must satisfy the inequalities

$$\begin{aligned}
2^{1.4} &< 2^{\sqrt{2}} < 2^{1.5}\\
2^{1.41} &< 2^{\sqrt{2}} < 2^{1.42}\\
2^{1.414} &< 2^{\sqrt{2}} < 2^{1.415}\\
2^{1.4142} &< 2^{\sqrt{2}} < 2^{1.4143}\\
2^{1.41421} &< 2^{\sqrt{2}} < 2^{1.41422}
\end{aligned}$$

and so on.

We replace the rational powers of 2 appearing in the last set of inequalities by appropriate decimal approximations and arrive at the following estimates for $2^{\sqrt{2}}$:

$$\begin{aligned}
2.639 < 2^{1.4} &< 2^{\sqrt{2}} < 2^{1.5} < 2.829\\
2.657 < 2^{1.41} &< 2^{\sqrt{2}} < 2^{1.42} < 2.676\\
2.664 < 2^{1.414} &< 2^{\sqrt{2}} < 2^{1.415} < 2.667\\
2.665 < 2^{1.4142} &< 2^{\sqrt{2}} < 2^{1.4143} < 2.666
\end{aligned}$$

and so on. Thus, if (1) is to hold, we know that, to 3 decimal places, $2^{\sqrt{2}} = 2.665 \ldots$.

The pinching down process that we use to estimate $2^{\sqrt{2}}$ is indicated in Figure 5-4b.

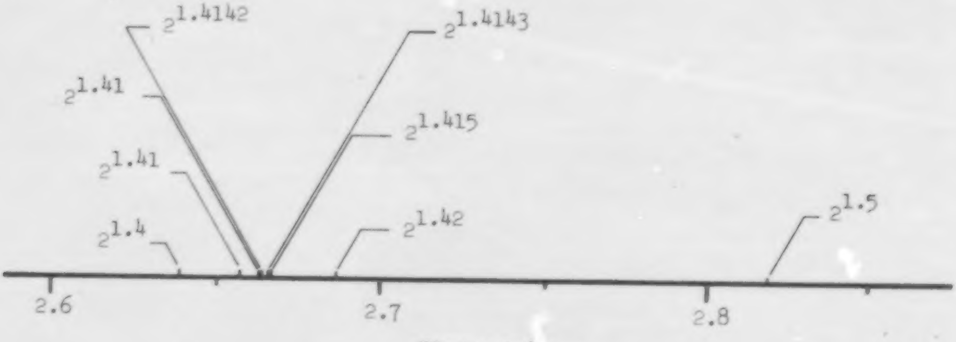


Figure 5-4b

Pinching down on $2^{\sqrt{2}}$.

To generalize to any real number x, we choose any increasing sequence $r_1, r_2, r_3, \ldots, r_n, \ldots,$ of rational numbers all less than x and any decreasing sequence $s_1, s_2, s_3, \ldots, s_n, \ldots,$ of rational numbers all greater than x such that the difference $s_n - r_n$ can be made arbitrarily small.

We compute the sequence of numbers $2^{r_1}, 2^{r_2}, 2^{r_3}, \ldots, 2^{r_n}, \ldots,$ and the sequence of numbers $2^{s_1}, 2^{s_2}, 2^{s_3}, \ldots, 2^{s_n}, \ldots,$ and then look at the intervals $2^{r_n} \leq y^{s_n}$.

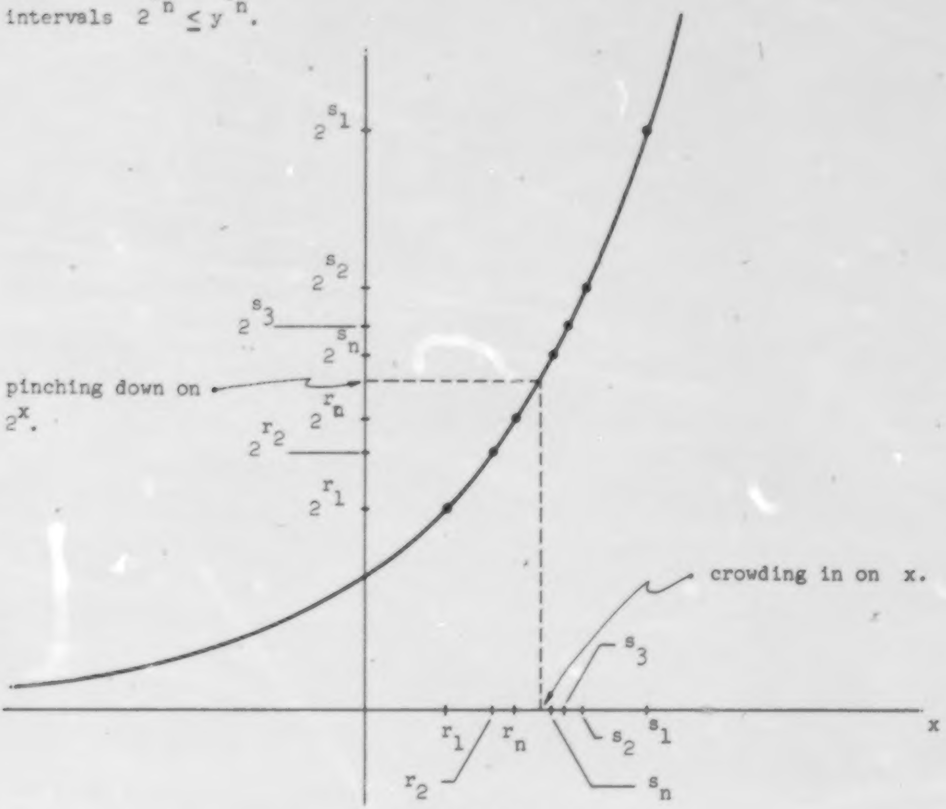

2^{s_1}
2^{s_2}
2^{s_3}
2^{s_n}
pinching down on 2^x.
2^{r_n}
2^{r_2}
2^{r_1}
crowding in on x.
s_3
r_1
r_n
s_2
s_1
x
r_2
s_n

Figure 5-4c

Pinching down on 2^x.

It is a property of the real number system that as x is confined by s_n and r_n to successively smaller intervals, the corresponding intervals on the y-axis pinch down to a uniquely determined number, which we shall define as the number 2^x. The number obtained is independent of the particular choice of the sequences $r_1, r_2, r_3, \ldots, r_n, \ldots$ and $s_1, s_2, s_3, \ldots, s_n, \ldots$.

A careful graph of $x \to 2^x$ is sketched in Figure 5-4d.

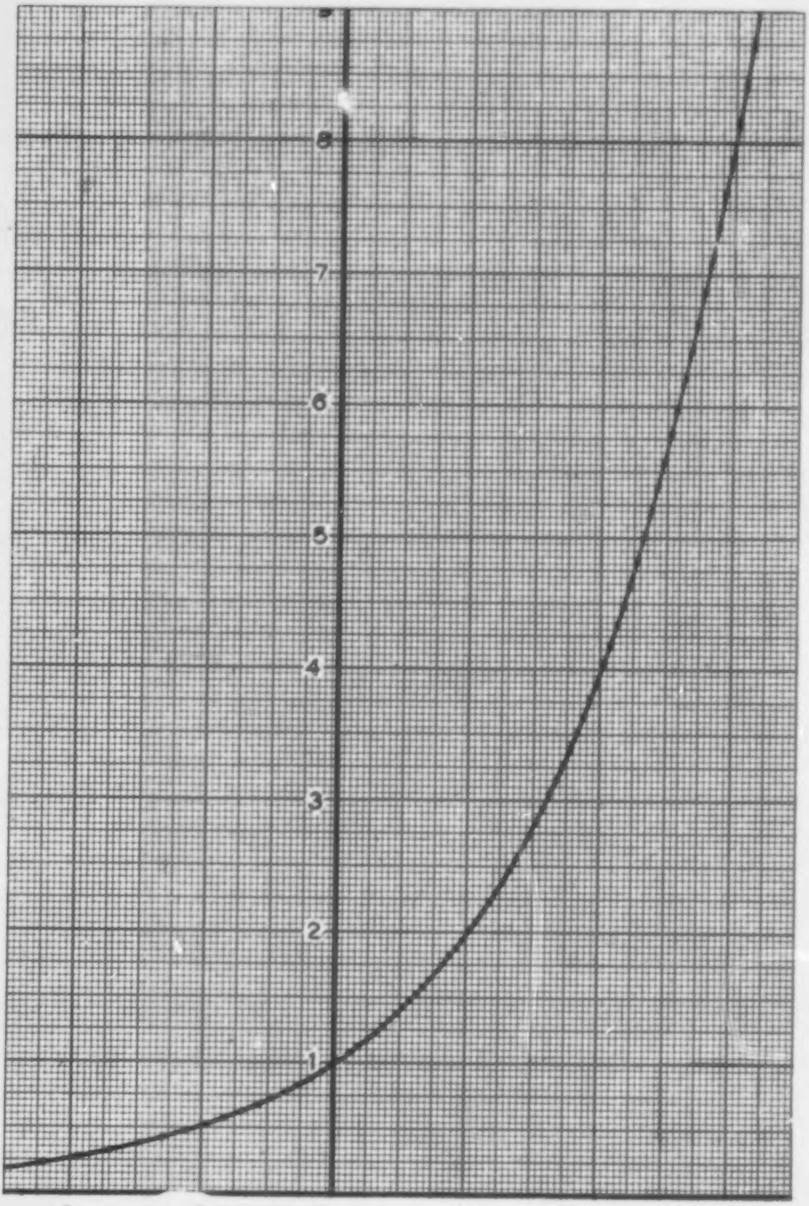


Figure 5-4d

$f : x \to 2^x$

Exercises 5-4

1. Use the graph of $x \rightarrow 2^x$ to estimate the value of:
   - (a) $2^{1.15}$
   - (b) $2^{2.65}$
   - (c) $2^{0.58}$
   - (d) $2^{-0.72}$

2. Compare your results in Number 1 with your answers to Number 2 in Exercises 5-3.

3. Use the graph of $x \rightarrow 2^x$ to estimate the value of:
   - (a) $2^{\sqrt{3}}$
   - (b) 2^{π}
   - (c) $2^{-\pi/4}$

4. Is there any value of x for which $2^x = 0$? Give reasons for your answer.

5. Use the graph of $x \rightarrow 2^x$ to estimate the value of x if:
   - (a) $2^x = 6$
   - (b) $2^x = 0.4$
   - (c) $2^x = 3.8$
   - (d) $2^x = 3$
   - (e) $2^x = 2.7$

5-5. Powers of the Base a as Powers of 2

We have concentrated on the function

$$f : x \to 2^x.$$

You are now familiar with its graph and have worked with a table of its values.

We shall now study the function

$$f : x \to a^x$$

where a is any positive real number. Fortunately we do not have to start from scratch because we can express a as a power of 2, as we proceed to show.

The graph of $f : x \to 2^x$ lies above the x-axis and rises from left to right. Also, $f(x) = 2^x$ becomes arbitrarily large for x sufficiently far to the right on the real number line, and arbitrarily close to zero for all x sufficiently far to the left on the real line. The graph has no gaps. Consequently, if we proceed from left to right along the graph, 2^x increases steadily in such a way that any given positive number a will be encountered once and only once. That is, there must be one and only one value of x, say α, for which

(1) $$2^\alpha = a$$

(See Figure 5-5a) and therefore a may be expressed as a power of 2.

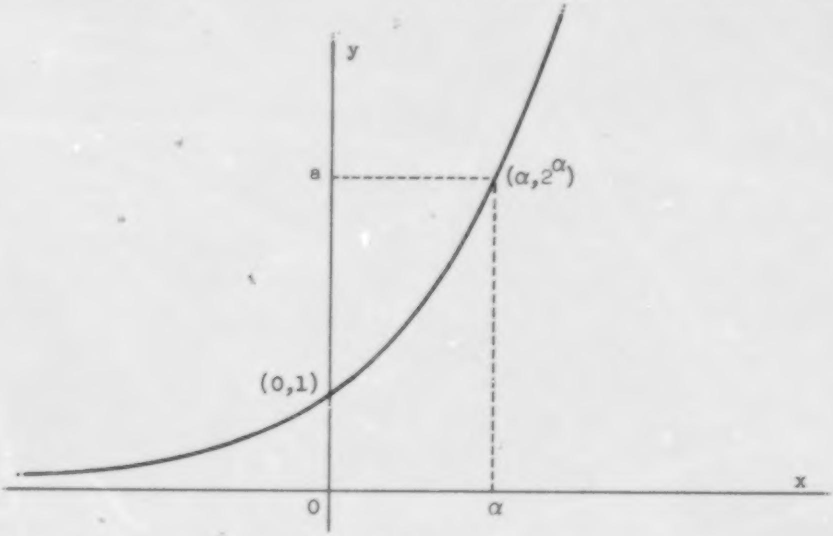


Figure 5-5a

Graph of $x \to 2^x$ showing that $2^\alpha = a$.

We can find the value of α by means of the graph (Figure 5-4d) or Table 5-3.

Example 5-5a. Find the value of α for which $1.11 = 2^{\alpha}$.

We look for 1.11 in the second column and read backward to find the corresponding value of α in the first column. Thus, $1.11 = 2^{0.15}$ (approximately).

Example 5-5b. Express 3.25 in the form 2^{α}.

We have $3.25 = 2(1.625) \approx 2^{1}(2^{0.70}) = 2^{1.70}$.

Example 5-5c. Find the value of α for which $2^{\alpha} = 6$.

On the graph of $x \to 2^{x}$ we look for the abscissa corresponding to the ordinate 6. The result is 2.6 (approximately).

If we use Table 5-3 to express 6 as a power of 2, we first write $6 = 2^{2}(1.5)$. Interpolating in Table 5-3 between the entries for $x = 0.55$ and 0.60 we obtain $2^{0.58} \approx 1.50$. Hence, $6 = 2^{2}(1.50) \approx 2^{2}(2^{0.58}) \approx 2^{2.58}$. Therefore $\alpha \approx 2.58$, by interpolation.

The expression a^{x} for x irrational and $a \neq 2$ has not yet been defined. We could follow the procedure of Section 5-4 to assign meaning to a^{x} when x is irrational. Since we can write $a = 2^{\alpha}$ we can simply define the function $x \to a^{x}$ by

(2) $$a^{x} = 2^{\alpha x} \quad \text{where} \quad a = 2^{\alpha}.$$

The laws of exponents will hold

$$a^{x+y} = a^{x}a^{y} \quad \text{and} \quad (a^{x})^{y} = a^{xy}$$

and the graph of $x \to a^{x}$ will have no gaps. These are consequences of (2), as is the fact that

(3) If $a > 1$, then $x \to a^{x}$ is increasing.

To prove (3), for example, note that if $a > 1$ then $a = 2^{\alpha}$ where $\alpha > 0$, (for if $\alpha \leq 0$ then $2^{\alpha} \leq 1$). Thus if $x < y$ then $\alpha x < \alpha y$ so that $2^{\alpha x} < 2^{\alpha y}$; that is, $a^{x} < a^{y}$.

The graph of $x \to a^x$ is obtained from the graph of $x \to 2^x$ by changing scale by the factor α, where $a = 2^{\alpha}$. For example, if $a = 4$ so that $\alpha = 2$, then we just "shrink" the x scale by a factor of 2. If $0 < a < 1$ and $a = 2^{\alpha}$, then α will be negative. In this case the graph of $x \to a^x$ is obtained by changing scale in $x \to 2^x$ and reflecting the graph in the vertical axis. Three cases are illustrated in Figure 5-5b. These considerations will be useful in our subsequent discussion (Chapter 6) of tangent lines to graphs of exponential functions.

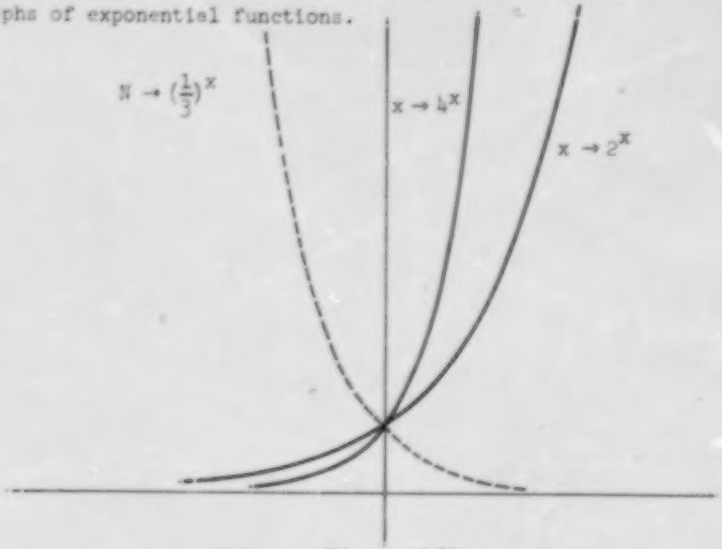


Figure 5-5b

The following examples illustrate the use of formula (2) Table 5-3 in calculating a^x.

Example 5-5d. Express $3^{0.7}$ as a power of 2, and find the approximate value of $3^{0.7}$.

To find the value of $3^{0.7}$ we first express 3 as a power of 2. Thus, $3 = 2^1(1.5) = 2^1(2^{0.58}) = 2^{1.58}$ (approximately). (Verify this from Figure 5-4d.)

Now
$$\begin{aligned} 3^{0.7} &\approx (2^{1.58})^{0.7} = 2^{1.106} \approx 2^{1.11} \\ &\approx 2^{(1+0.10+0.01)} = 2^1(2^{0.10})(2^{0.01}) \\ &\approx 2(1.072)(1.007) \approx 2.159. \end{aligned}$$

Example 5-5e. Calculate the value of $(6.276)^{0.4}$.

We note that

$$6.276 = 4(1.569) = 2^2(1.569) \approx 2^2(2^{0.65}) = 2^{2.65}$$

(Verify this from Figure 5-4d.) Hence,

$$\begin{aligned}(6.276)^{0.4} &\approx (2^{2.65})^{0.4} = 2^{(2.65)(0.4)} = 2^{1.06}\\ &\approx 2^{(1+0.05+0.01)} = 2^1 \cdot 2^{0.05} \cdot 2^{0.01}\\ &\approx 2(1.035)(1.007) = 2.084 \text{ (approximately).}\end{aligned}$$

Exercises 5-5

1. Express 3.4 in the form 2^α.
2. Write 2.64 in the form $2.64 = 2^\alpha$ and then find the approximate value of $(2.64)^{0.3}$.
3. Find the approximate value of $(6.276)^{-0.6}$.
4. Find the approximate value of $(5.2)^{2.6}$.
5. Show that if $0 < a < 1$ and $v > u$, then $a^v < a^u$.
6. By finding a suitable value of α, express each of the following functions in the form of

$$x \to 2^{\alpha x}.$$

   (a) $x \to 4^x$

   (b) $x \to (3.60)^x$

   (c) $x \to (5.736)^x$

   (d) $x \to (0.420)^x$

7. Suppose a and b are positive and different from 1. Consider the two functions:

$$x \to a^x \quad \text{and} \quad x \to b^x$$

which can be respectively written as

$$x \to 2^{\alpha x} \quad \text{and} \quad x \to 2^{\beta x}.$$

(a) If $a < b$, what is the relationship between α and β.

(b) Suppose a is close to b. Is α close to β? Illustrate your answer by completing the following table:

	For these values of b			
	4	3	2.3	2.04
β =				

	a =
	2
α =	

8. Given: $a^x = 2^{\alpha x}$.

(a) If $x \to a^x$ is increasing, what is the sign of α?

(b) If $x \to a^x$ is decreasing, what is the sign of α?

(c) Show that $\dfrac{f(x+1)}{f(x)}$ is independent of x.

(i) If $\alpha > 0$, what can we conclude about the above quotient?

(ii) If $\alpha < 0$, what can we conclude about the above quotient?

(d) If $\alpha > 2$, show that $a^x > (2^x)^2$ for $x > 0$.

9. (a) Where does the graph of $x \to a^x$, $(a > 0)$ cross the y-axis? Does your answer depend upon a?

(b) Find the point(s) of intersection, if any, of the two graphs:

$$x \to a^x \quad \text{and} \quad x \to 2(2a)^x.$$

(c) Find the point(s) of intersection, if any, of the two graphs:

$$x \to a^x \quad \text{and} \quad x \to b(ba)^x$$

(where b is a real number greater than zero).

(d) Find the point(s) of intersection, if any, of the two graphs:

$$x \to a^x \quad \text{and} \quad x \to b^n(ba)^x$$

(where b and n are real numbers greater than zero).

5-6. The Logarithm (Base 2)

Since the exponential function $x \to 2^x$ is increasing, its graph crosses every horizontal line $y = b$ where $b > 0$. Therefore, the inverse function is increasing, its graph has no gaps, it becomes arbitrarily large as x becomes large and it comes arbitrarily close to the x-axis for x negative and $|x|$ large. In particular, if $x > 0$, there is exactly one real number y such that

$$2^y = x.$$

This number y is called the "logarithm of x to the base 2" and is denoted by $\log_2 x$. Thus the function

$$\log_2 : x \to \log_2 x$$

is defined only for $x > 0$ and is the inverse of the exponential function with base 2. These two functions are related by

(1) $$\log_2(c) = d \text{ if and only if } 2^d = c.$$

In terms of graphs, this tells us that

(2) If (c,d) lies on the graph of $x \to \log_2 x$ then (d,c) lies on the graph of $x \to 2^x$ and conversely.

As was the case for other functions and their inverses that we have studied, the graph of $x \to \log_2 x$ can be obtained by folding the graph of $x \to 2^x$ over the line given by $y = x$. (See Figure 5-6a.)

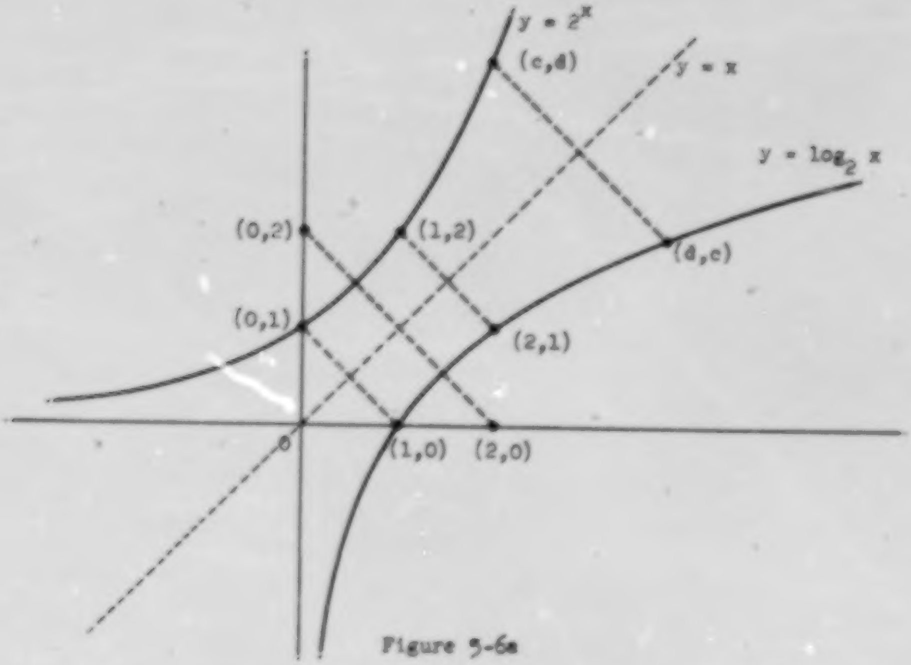


Figure 5-6a

Calculations involving $\log_2$ can be carried out using the relation (1). For example, since

$$2^3 = 8, \text{ we know that } \log_2 8 = 3;$$

and since

$$2^{-6} = \frac{1}{64}, \text{ we have } \log_2(\frac{1}{64}) = -6.$$

In fact, any table of values of the exponential function to the base 2 will also give values of $\log_2$. For example, Table 5-3 gives

$$2^{.20} = 1.14870$$

so that

$$\log_2 1.14870 = 0.20.$$

A number of useful properties of $\log_2$ can be derived from properties of the exponential function by using the relation (1). Some of these are

(a) $\log_2 1 = 0$

(b) $\log_2 2 = 1$

(3) (c) $\log_2$ is an increasing function

(d) $\log_2 x > 0$ if $x > 1$

(e) $\log_2 x < 0$ if $0 < x < 1$.

The properties can be observed in Figure 5-6a. They can also be proved. For example, let us prove (c). Suppose $x_0 < x_1$. Put $y_0 = \log_2 x_0$ and $y_1 = \log_2 x_1$ so that

$$x_0 = 2^{y_0} \text{ and } x_1 = 2^{y_1}.$$

If y_0 were not less than y_1 we would have $y_0 \geq y_1$. Since $x \to 2^x$ is an increasing function, the condition $y_0 \geq y_1$ implies that

$$2^{y_0} \geq 2^{y_1};$$

that is, $x_0 \geq x_1$. This contradicts the assumption that $x_0 < x_1$ so we are forced to conclude that $\log_2 x_0 < \log_2 x_1$. This proves that $\log_2$ is an increasing function.

The laws of exponents

$$2^{x+y} = 2^x 2^y \text{ and } (2^x)^y = 2^{xy}$$

give rise to the following logarithm laws:

(4) (a) $\log_2 xy = \log_2 x + \log_2 y$

(b) $\log_2 x^y = y \log_2 x$.

For example, to prove the first of these (4a) we let

(5) $a = \log_2 xy, \quad b = \log_2 x, \quad c = \log_2 y$

so that

$$xy = 2^a, \quad x = 2^b, \quad y = 2^c.$$

Observe that xy is then also given by

$$xy = 2^b \cdot 2^c = 2^{b+c}$$

so that

$$2^a = 2^{b+c}.$$

We conclude that $a = b + c$; that is (from (5)),

$$\log_2 xy = \log_2 x + \log_2 y.$$

Formula (4b) is left to Exercises 5-6, Number 3. These formulas can be used with Table 5-3 to calculate logarithms of numbers not appearing in the table.

Example 5-6a. Find $\log_2 3.25$.

Upon looking at the second column of Table 5-3 we see that 3.25 doesn't appear. Note however that

$$3.25 = 2(1.625)$$

so that

$$\log_2 3.25 = \log_2 2 + \log_2 1.625.$$

Reading from the second column to the first column in Table 5-3 we obtain

$$\log_2 2 = 1, \quad \log_2 1.625 \approx 0.70,$$

so that

$$\log_2 3.25 \approx 1.70.$$

The next example shows how inequalities for the logarithm to the base 2 are obtained from inequalities for the exponential.

Example 5-6b. Show that if n is a positive integer then

$$\log_2 n < n.$$

Since

$$2^n > n$$

and $\log_2$ is an increasing function we must have

$$\log_2 2^n > \log_2 n.$$

Formula (4b) gives

$$\log_2 2^n = n \log_2 2.$$

Since $\log_2 2 = 1$ we must have

$$n > \log_2 n.$$

Example 5-6c. Sketch the graph of $f : x \to \log_2(1 - 2x)$.

The function f is defined for $1 - 2x > 0$; that is, for $x < \frac{1}{2}$. We can write $\log_2(1 - 2x) = \log_2 [-2(x - \frac{1}{2})]$. The graph of $f : x \to \log_2 [-2(x - \frac{1}{2})]$ can be obtained from the graph of $g : x \to \log_2 x$ in four steps. (See Figure 5-6b.)

First, we begin with the graph of g.
Second, replace x by $2x$.
Third, fold the graph over the line given by $x = 0$.
(the y-axis)

Fourth, shift the graph one-half unit to the right.

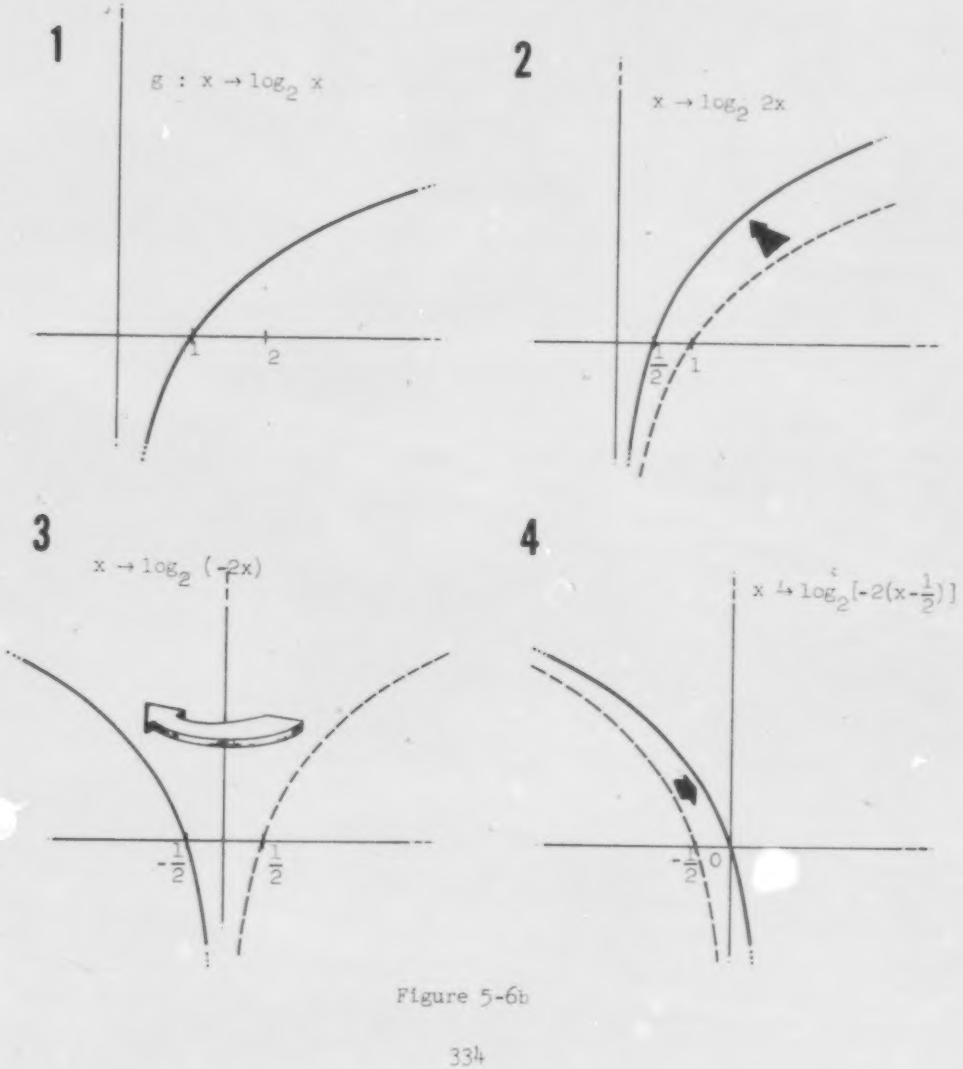

1
$g : x \to \log_2 x$
1
2
2
$x \to \log_2 2x$
$\frac{1}{2}$
1
3
$x \to \log_2 (-2x)$
$-\frac{1}{2}$
$\frac{1}{2}$
4
$x \to \log_2[-2(x-\frac{1}{2})]$
$-\frac{1}{2}$
0

Figure 5-6b

Exercises 5-6

1. Prove that for any real number $x > 0$, $\log_2(x \cdot \frac{1}{x}) = 0$, and hence that $\log_2(\frac{1}{x}) = -\log_2 x$.

2. Prove that for any real numbers $x_1 > 0$, $x_2 > 0$, $\log_2 (\frac{x_1}{x_2}) = \log_2 x_1 - \log_2 x_2$.

3. Prove formula 4(b): $\log_2 x^y = y \log_2 x$

4. (a) For what values of x is $\log_2 x$ less than 0? greater than 0?
   (b) For what values of x is y less than 0? greater than 0?
   - (i) $y = \log_2 2x$
   - (ii) $y = \log_2 (-x)$
   - (iii) $y = \log_2 (x - 1)$
   - (iv) $y = \log_2 (1 - x)$
   - (v) $y = \log_2(2x - 3)$
   - (vi) $y = \log_2 (3 - 2x)$

5. Graph the following functions on one set of axes over the interval $-5 \leq x \leq 5$.
   - (a) $x \rightarrow \log_2 x$
   - (b) $x \rightarrow \log_2 (-x)$
   - (c) $x \rightarrow \log_2 (x - 1)$
   - (d) $x \rightarrow \log_2 (1 - x)$
   - (e) $x \rightarrow \log_2 (x + 1)$
   - (f) $x \rightarrow \log_2 [-(1 + x)]$

6. On one set of axes sketch the graph of each of the following functions over the interval $0 \leq x \leq 8$.
   - (a) $x \rightarrow \log_2 x$
   - (b) $x \rightarrow \log_2 (2x)$
   - (c) $x \rightarrow \log_2 (-2x)$
   - (d) $x \rightarrow \log_2 (\frac{x}{2})$
   - (e) $x \rightarrow \log_2 (-\frac{x}{2})$

7. On one set of axes sketch the graph of each of the following functions over the interval $-1 \leq x \leq 8$.
   - (a) $x \rightarrow \log_2 2x$
   - (b) $x \rightarrow \log_2 (2x - 2)$
   - (c) $x \rightarrow \log_2 (2x - 7)$
   - (d) $x \rightarrow \log_2 (7 - 2x)$

8. Write each logarithmic statement in exponential form:

(a) $\log_2 a = b$

(b) $3 \log_2 x = 5k$

(c) $\log_2 z + \log_2 w = 4$

(d) $-2 \log_2 3 = -t$

9. Write each exponential statement in base 2 logarithmic form:

(a) $2^x = 13$

(b) $4^k = \frac{3}{5}$

(c) $(\frac{1}{2})^{-3} = mn$

(d) $2\sqrt{2} = x^y$

10. Evaluate each of the following:

(a) $\log_2 2$

(b) $\log_2 4$

(c) $\log_2 8$

(d) $\log_2 16$

(e) $\log_2 1$

(f) $\log_2 \frac{1}{2}$

(g) $\log_2 \frac{1}{4}$

(h) $\log_2 \frac{1}{8}$

11. Using the results (and extensions) of Number 10 above, but without the use of tables, locate the values of each of the following between consecutive integers (e.g., $3 < \log_2 11 < 4$, since $2^3 < 11 < 2^4$).

(a) $\log_2 3$

(b) $\log_2 5$

(c) $\log_2 6$

(d) $\log_2 7$

(e) $\log_2 9$

(f) $\log_2 10$

(g) $\log_2 13$

(h) $\log_2 18$

(i) $\log_2 31$

(j) $\log_2 34$

(k) $\log_2 60$

(ℓ) $\log_2 99$

(m) $\log_2 (0.9)$

(n) $\log_2 (\frac{3}{4})$

(o) $\log_2 (\frac{3}{8})$

(p) $\log_2 (0.18)$

12. In evaluating the following, first estimate your answer from your solutions to Number 11, then make a closer estimate from the use of Table 5-3.

(a) $\log_2 3$

(b) $\log_2 5$

(c) $\log_2 13$

13. Using the results of Numbers 10 and 12, estimate (e.g., $\log_2 72 = \log_2(2^3)(3^2) = 3 \log_2 2 + 2 \log_2 3 \approx 6.28$) each of the following.

(a) $\log_2 6$

(b) $\log_2 12$

(c) $\log_2 24$

(d) $\log_2 9$

(e) $\log_2 27$

(f) $\log_2 169$

(g) $\log_2 54$

(h) $\log_2 36$

(i) $\log_2 52$

14. Find a value for x which makes the following assertions true.

(a) $\log_2 x = 0$

(b) $\log_2 x = 4$

(c) $\log_2 x = -1$

(d) $\log_2 x = \frac{1}{2}$

(e) $\log_2 x = -\frac{1}{2}$

(f) $\log_2 x = \frac{3}{4}$

15. Show that if $n > 1$, then

$$\log_2 n(n - 1) < n + 1.$$

(Hint: Use the fact that $2^n > \frac{n(n - 1)}{2}$.)

5-7. Logarithms (General Base)

If $a > 0$ and $a \neq 1$ then the logarithm to the base a, denoted by $\log_a$, is defined analogously to $\log_2$. Thus $\log_a x$ is defined only for $x > 0$ and is given by

$$\log_a x = y \quad \text{if} \quad a^y = x. \tag{1}$$

The techniques and ideas of the previous section extend easily to this more general case. For example, the graph of

$$x \to \log_a x$$

is obtained by folding the graph of $x \to a^x$ over the line given by $y = x$.

We summarize other easily obtained properties:

(2)

(a) $$\log_a 1 = 0$$

(b) $$\log_a a = 1$$

(c) if $a > 1$, $\log_a$ is an increasing function and if $0 < a < 1$ then $\log_a$ is decreasing.

The laws of exponents

$$a^{x+y} = a^x a^y, \quad (a^x)^y = a^{xy}$$

give the corresponding logarithm formulas

(3)

(a) $\log_a xy = \log_a x + \log_a y$, for $x > 0$ and $y > 0$

(b) $\log_a x^y = y \log_a x$, for $x > 0$.

Logarithms to the base 10 are very useful in calculations, due to the fact that our number system is the decimal system. Logarithms to the base 2 are of increasing importance due to the use of the binary system in computors and in information theory. In the next chapter, our discussions of tangent lines will establish the importance of the base e, where e is the symbol for the (irrational) number, which correct to two places, is given by 2.72.

To convert from one base to another the following formula is useful. If a, b and c are each positive and unequal to 1 then

$$\log_a b = \frac{\log_c b}{\log_c a}. \tag{4}$$

To prove this we let $x = \log_a b$ so that

$$a^x = b.$$

Now take the logarithm to the base c of each side. This gives

$$\log_c a^x = \log_c b.$$

Formula 3(b) gives

$$\log_c a^x = x \log_c a$$

so that

$$x \log_c a = \log_c b.$$

Since $\log_c a \neq 0$ we can divide by it to obtain (4).

Another formula of interest is

If a and b are both positive and not equal to 1 then

$$\log_a b = \frac{1}{\log_b a}\ .$$

The proof of this is left to Exercises 5-7, Number 18.

Exercises 5-7

1. Write each expression in simpler form.
   (a) $a^{\log_a 3}$
   (b) $a^{2 \log_a 3}$
   (c) $a^{1/2 \log_a 3}$
   (d) $8^{\log_2 5}$
   (e) $16^{\log_4 2}$
   (f) $32^{\log_2 \sqrt[5]{2}}$

2. What is the value of x if $32 = 4^x$?

3. If $a \cdot a^m = (a^2)^m$, what is the value of m?

4. Prove that for x any real number > 0, $\log_a(x \cdot \frac{1}{x}) = 0$, and hence $\log_a(\frac{1}{x}) = -\log_a x$.

5. Prove that $\log_a (\frac{x_1}{x_2}) = \log_a x_1 - \log_a x_2$.

6. Show that $\log_a a = 1$. Write this equation in exponential form.

7. Express in exponential form

   (a) $\log_{10} 35 = y$

   (b) $\log_2 25 = x$

   (c) $2 \log_{10} 5 = x$

8. Given $\log_{10} 2 = 0.3010$ find

   $\log_{10} 5$, $\log_{10}(\frac{1}{2})$, $\log_{10}(\frac{25}{4})$, $\log_{10}(\frac{128}{5})$.

9. Express each of the following in logarithmic form.

   (a) $\sqrt[3]{125} = 5$

   (b) $10^{-2} = 0.01$

   (c) $27^{4/3} = 81$

   (d) $0.04^{3/2} = 0.008$

   (e) $\sqrt{\sqrt{16}} = 2$

10. Solve for x,

$$\log_6 (x + 9) + \log_6 x = 2$$

11. Express each of the following logarithms in terms of r, s, and t, if $r = \log_{10} 2$, $s = \log_{10} 3$, $t = \log_{10} 5$.

   (a) $\log_{10} 4$

   (b) $\log_{10} 6$

   (c) $\log_{10} \frac{1}{8}$

   (d) $\log_{10} 10$

   (e) $\log_{10} 2.5$

   (f) $\log_{10} \frac{2}{9}$

   (g) $\log_{10} \frac{5}{9} \sqrt{3}$

   (h) $\log_{10} 8 \sqrt[3]{100}$

12. Write the following logarithms as numbers.

(a) $\log_{10} 1000$

(b) $\log_{0.01} 0.001$

(c) $\log_3 (\frac{1}{81})$

(d) $\log_4 32$

(e) $\log_{10} (0.0001)$

(f) $\log_{0.5} 16$

(g) $\log_2 2^3$

(h) $\log_{10} \sqrt{10}$

(i) $\log_{81} 27$

(j) $\log_2 \sqrt{32}$

13. In each case determine the value of x.

(a) $4^{\log_4 5} + 3^{\log_3 5} = 2^{\log_2 x}$

(b) $\log_{10}(x^2 - 1) - 2 \log_{10} (x - 1) = \log_{10} 3$

(c) $7^{\log_x 5} = 5$

14. Solve the following equations.

(a) $\log_{10} x = 0$

(b) $\log_{10} x + 1 = 0$

(c) $\log_{10} x = 1$

(d) $\log_{10} (x - 2) = 3$

(e) $\log_{10} x + 3 = 0$

(f) $\log_{10} (2x - 1) + 2 = 0$

15. For what value(s) of x does it hold that

(a) $\log_c x = 0$

(b) $\log_x x = 1$

(c) $x^{\log_x c} = c$

(d) $\log_x 2^x = 2$

16. (a) Show that if $a > 1$, then $x \rightarrow \log_a x$ is an increasing function.

(b) Show that if $0 < a < 1$, then $x \rightarrow \log_a x$ is a decreasing function.

17. (a) How are the graphs of $x \rightarrow \log_a x$ and $x \rightarrow \log_b x$ related? (Hint: Use (4).)

(b) At what point does the graph of $x \rightarrow \log_a x$ cross the x-axis?

18. Prove formula (5) of the text: if a and b are both positive and not equal to 1, then

$$\log_a b = \frac{1}{\log_b a} .$$

Appendix 1

FUNCTIONS AND THEIR REPRESENTATIONS

A1-1. Functions

The precise definition of function can be formulated in many ways: as a set of ordered pairs (usually, ordered pairs of numbers), as an association or correspondence between two sets, etc. But no matter what definition we choose for a function, three things are required: a set called its domain, a set called its range, and a way of selecting a member of the range for each member of the domain.

Example A1-1a. The multiplication of integers by 2 defines a function. The domain of this function is the set of all integers; the range of the function is the set of all even integers.

We choose to define a function as an association between elements of two sets; thus the function of Example A1-1a associates with each integer its double.

> If with each element of a set A there is associated exactly one element of a set B, then this association is called a function, from A to B. The set A is called the domain of the function, and the set C of all members of B assigned to members of A by the function is called the range of the function.

In what follows we shall be exclusively concerned with functions whose domains are subsets of real numbers and whose ranges are also subsets of real numbers. More complicated functions (like "vector valued functions") may be built from these.

The range C may be the whole set B, in which case the function is called an onto function, or it may be a proper subset of B. In any case, we generally take for B the whole set of reals, because a function is usually specified before its range is considered.

It is common practice to represent a function by the letter f (other letters such as F, g, h, ∅, etc, will also be used). If x is an element of the domain of a function f, then f(x) denotes the element of the range which f associates with x. (Read for f(x) "the value of the function f at x," or simply "f at x," or "f of x.") An arrow is used to suggest the association of f(x) with x:

$$f : x \longrightarrow f(x)$$

(read "f takes x into f(x)"). This notation tells us nothing about the function f or the element x; it is merely a symbolic description of the relation between x and f(x).

Example A1-1b. Consider a function f defined as follows: f takes each number of the domain into its square. Thus, if 3 is an element of the domain, then f takes 3 into 9, or f associates 9 with 3. Concisely f(3) = 9. In general, if x represents any number in the domain of f, then f takes x into x^2:

$$f : x \longrightarrow x^2 \quad \text{or} \quad f(x) = x^2.$$

The function is not adequately defined until we specify its domain. If the domain is the set of all integers {..., -2, -1, 0, 1, 2, ...}, then the range is a subset of nonnegative integers, {0, 1, 4, 9, 16, ...}. If we choose the set of all real numbers as domain, then a different function is defined, even though the rule of association is the same; in this case the range of the function is the set of nonnegative real numbers.

Observe that a function from A to B is a one-way association; the reverse association from B to A is not necessarily a function. In Example A1-1b, f(3) = 9 and f(-3) = 9, while the reverse association would assign both 3 and -3 to 9, violating the definition of a function.

It is often useful to think of a function as a mapping, and we say that a function maps each element of its domain upon one and only one element of its range. In this vein, $f : x \longrightarrow f(x)$ can be read, "f maps x upon f(x)"; f(x) is called the image of x under the mapping, and x is called a preimage of f(x). This notion is illustrated in Figure A1-1a, where elements of the domain A and range B are represented by points and the mapping is suggested by arrows from the points of the domain to corresponding points of the range.

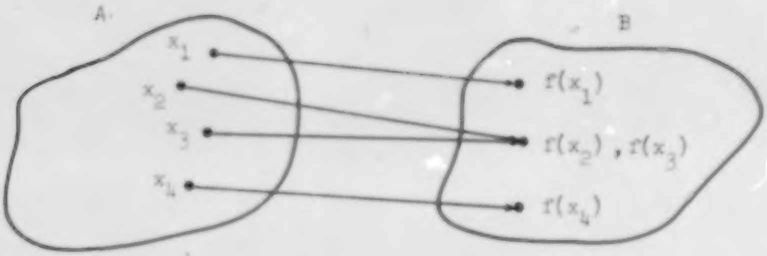


Figure A1-1a

Note that each element of the domain is mapped into a unique element of the range; i.e., each arrow starts from a different point in the domain. This is the requirement of our definition, that with each element of the domain there is associated exactly one element of the range.

Our definition of function contains the rather vague phrase, "there is associated." The manner of association must be specified whenever we are dealing with a particular function. In this course, a function will generally be defined by a formula giving its value: for example, $f(x) = 3x - 5$; $g(x) = x^2 + 3x + 7$. Other ways of defining a function include verbal description, graph, and table.

The notation $f(x)$ is particularly convenient when we refer to values of a function; i.e., elements in the range of the function. We illustrate this in the next example.

Example A1-1c. Consider the function

$$f : x \longrightarrow 3x^2 - 5$$

whose domain is the set of all real numbers. Then

$$f(x) = 3x^2 - 5,$$
$$f(-2) = 3(-2)^2 - 5 = 7,$$
$$f(0) = 3(0)^2 - 5 = -5,$$

and if $a + \sqrt{b}$ is a real number, then $f(a + \sqrt{b}) = 3(a + \sqrt{b})^2 - 5$.

We note, since x^2 may be any nonnegative real number, that $3x^2 - 5 \geq -5$, and hence the range of f is the set of all real numbers not less than -5.

As mentioned earlier, a function is not completely defined unless the domain is specified. If no other information is given, it is a convenient practice, especially when dealing with a function defined by a formula, to assume that the domain includes all real numbers for which the formula describes a real number. For example, if a domain is not specified for the function $f : x \longrightarrow \frac{2x}{x^2 - 9}$, then the domain is assumed to be the set of all real numbers except 3 and -3. Similarly, if g is a function such that $g(x) = \sqrt{4 - x^2}$, we assume, in the absence of any other information, that the domain is $\{x : -2 \leq x \leq 2\}$; that is, the set of all real numbers x from -2 to 2 inclusive.

We note here that two functions f and g are identical if and only if they have the same domain and $f(x) = g(x)$ for each x in their domain.

The graph of a function is perhaps its most intuitively illuminating representation; it conveys important information about the function at a glance. The graph of f is the set of all those points (x,y) for which x is in the domain of f and $y = f(x)$.

Example A1-1d. The graph of the function $f : x \longrightarrow y = \sqrt{25 - x^2}$ is the semicircle shown in Figure A1-1b.* The graph gives us a clear picture of what the function is doing to the elements of its domain, and we can, moreover, usually infer from the graph any limitations on the domain and range. Thus, it is easily determined from Figure A1-1b that the domain of f is the set of all x such that $-5 \leq x \leq 5$ and the range is the set of all y such that $0 \leq y \leq 5$. These sets are represented by the heavy segments on the x- and y-axis, respectively.

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* In this figure a complete graph is displayed. The graph in Figure A1-1c, as well as most of the graphs in the text, are necessarily incomplete.

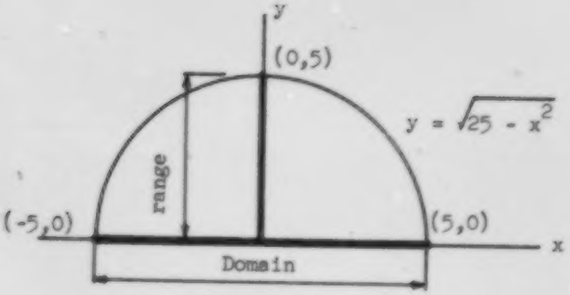

Figure A1-1b

We remind you of the fact that not every curve is the graph of a function. In particular, our definition requires that a function map each element of its domain onto only one element of its range. In terms of points of a graph, this means that the graph of a function does not contain the points (x_1,y_1) and (x_1,y_2) if $y_1 \neq y_2$; i.e., two points having the same abscissa but different ordinates. This is the basis for the "vertical line test": if in the xy-plane we imagine all possible lines which are parallel to the y-axis, and if any of these lines cuts the graph in more than one point, then the graph represents a relation which is not a function. Conversely, if every line parallel to the y-axis intersects a graph in at most one point, then the graph is that of a function.

Example A1-1e. The equation $x^2 + y^2 = 25$, whose graph is a circle with radius 5 and center at the origin, does not define a function. On the open interval $-5 < x < 5$, every value of x is associated with two different values of y, contrary to the definition of function. Specifically, $(3,4)$ and $(3,-4)$ are two points of the circle; they determine a line parallel to the y-axis and intersecting the circle in two points, thus illustrating that the circle is the graph of a relation that is not a function. We can, however, separate the circle into two semi-circles--the graphs of the functions $x \longrightarrow \sqrt{25 - x^2}$ (Example A1-1d) and $x \longrightarrow -\sqrt{25 - x^2}$.

Throughout this discussion we have used the letters x and y to represent elements of sets. Specifically, if f is the function

$$f : x \longrightarrow y = f(x),$$

then x represents an element (unspecified) in the domain of f, and y represents the corresponding element in the range of f. In many textbooks x and y are called variables, and since a particular value of y in the range depends upon a particular choice of x in the domain, x is called the *independent variable* and y the *dependent variable*. The functional relationship is then described by saying that "y is a function of x." For the most part this language is not used in this textbook.

We conclude this section with a summary of several different special functions; you are undoubtedly acquainted with some of them.

The Constant Function. If b is an arbitrary real number, then the function f which associates with every real number x the value b, $f : x \longrightarrow b$, is called a *constant function*. More generally, any function whose range contains exactly one number is a constant function. The graph of a constant function, say $f : x \longrightarrow c$ for all real x, is a line parallel to and $|c|$ units from the x-axis.

The Identity Function. Let A be the set of all real numbers; with each number a in A, associate the number a. This association defines a function whose domain is A and whose range is A, namely

$$f :: x \longrightarrow x.$$

More generally, for any domain such a function is called the *identity function*. If the domain is the set of all real numbers, then the graph of f is the line with equation $y = x$.

The Absolute Value Function. With each real number the *absolute value function* associates its absolute value:[*]

$$f : x \longrightarrow |x| = \begin{cases} x & \text{for } x \geq 0, \\ -x & \text{for } x < 0. \end{cases}$$

---

[*] Alternative definitions:

$$f : x \longrightarrow |x| = \max \{x,-x\};$$

$$f : x \longrightarrow |x| = \sqrt{x^2}.$$

The graph of f is shown in Figure A1-1c, it is the union of two rays issuing from the origin.

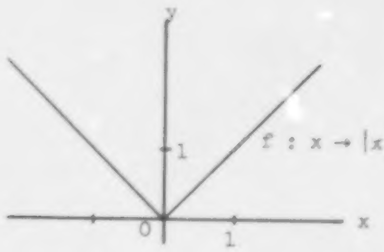

Figure A1-1c

The Integer Part Function.[*] Every real number x can be represented as the sum of an integer n and a real number r such that

$$x = n + r, \quad n \leq x, \quad \text{and} \quad 0 \leq r < 1.$$

For example,

$$\begin{aligned} 5.38 &= 5 + .38, \\ 3 &= 3 + 0, \\ -2.4 &= -3 + .6. \end{aligned}$$

We call n the integer part of x and denote it by $[x] = n$; it follows that $[x] \leq x < [x] + 1$. Thus we see that to each real number x there corresponds a unique integer part $[x]$, and this correspondence defines the integer part function

$$f : x \longrightarrow [x].$$

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[*] Sometimes called the greatest integer function.

A graph of this function is shown in Figure A1-1d; it is called a step graph; i.e., the graph of a step function.

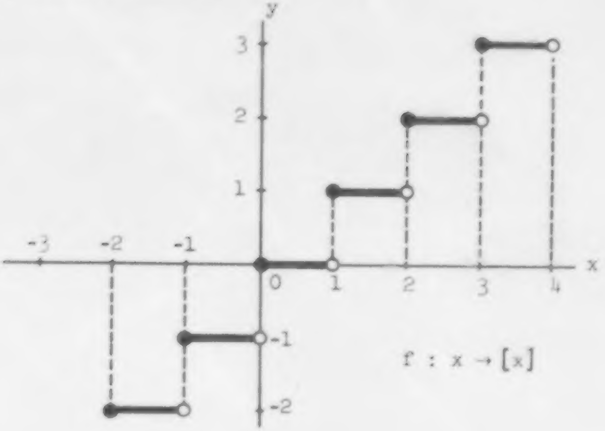

Figure A1-1d

The Signum Function. With each positive real number associate the number +1, with zero associate the number 0, and with each negative real number associate the number -1. These associations define the signum function, symbolized by sgn x. Thus

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

We leave it as an exercise for you to sketch the graph of this function.

Even and Odd Functions. Let f be a function whose domain contains $-x$ whenever it contains x. The function f is said to be even if $f(-x) = f(x)$. For example, the function f with values $f(x) = x^2$ is even since $(-x)^2 = x^2$ for all x. Geometrically the graph of an even function is symmetric with respect to the y-axis.

The function f is said to be odd if $f(-x) = -f(x)$. For example, the function f with values $f(x) = x^3$ is odd since $(-x)^3 = -x^3$ for all x. Geometrically the graph of an odd function is symmetric with respect to the origin.

Periodic Functions. Certain functions have the property that their function values repeat themselves in the same order at regular intervals over the domain (Figure A1-1e).

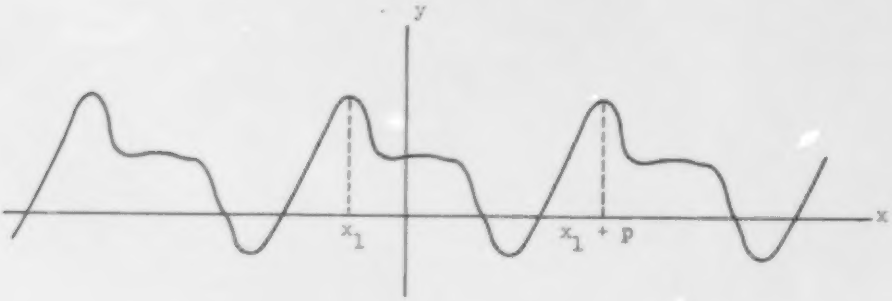

Figure A1-1e

Functions having this property are called periodic; included in this important class are the circular (trigonometric) functions, to be discussed in Chapter 2 and 3.

A function f is periodic and has period p, $p \neq 0$, if and only if, for all x in the domain of f, $x + p$ is also in the domain and

$$f(x + p) = f(x). \tag{1}$$

From the definition we note that each successive addition or subtraction of p brings us back to $f(x)$ again. For example,

$$\begin{aligned} f(x + 2p) &= f\big((x + p) + p\big) \\ &= f(x + p) \\ &= f(x), \end{aligned}$$

and

$$\begin{aligned} f(x - p) &= f\big((x - p) + p\big) \\ &= f(x). \end{aligned}$$

In general, we infer that any multiple of a period of f is also a period; that is,

$$f(x + np) = f(x) \qquad \text{for any integer } n.$$

For a constant function

$$f : x \longrightarrow c,$$

it is obvious that f is periodic with any period p, since

$$f(x + p) = c = f(x).$$

It can be shown that for nonconstant periodic functions (continuous at one point at least) there is a least positive value of p for which (1) is true. This is called the *fundamental period*, or simply *the period*, of such a function.

Example A1-1f. $f : x \longrightarrow x - [x]$, x real, is a periodic function. If $x = n + r$ where n is the integer part of x and r its fractional part,[*] then

$$\begin{aligned} f(x) &= f(n + r) \\ &= (n + r) - [n + r] \\ &= n + r - n \\ &= r, \end{aligned}$$

and

$$\begin{aligned} f(x + 1) &= f(n + 1 + r) \\ &= (n + 1 + r) - [n + 1 + r] \\ &= n + 1 + r - (n + 1) \\ &= r. \end{aligned}$$

Thus, as was asserted, f is periodic and its period is 1, as shown in its graph (Figure A1-1f).

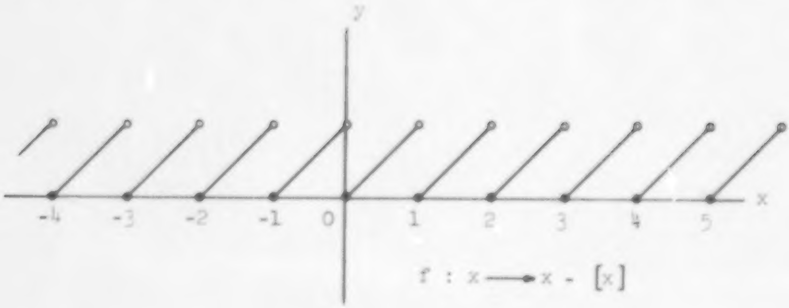

Figure A1-1f

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[*] We note that since $f(x) = r$, the fractional part of x, this function is sometimes called the fractional part function.

Exercises A1-1

1. Below are given examples of associations between elements of two sets. Decide whether each example may properly represent a function. This also requires you to specify the domain and range for each function. Note that no particular variable has to be the domain variable, and also that some of the relations may give rise to several functions.

   (a) Assign to each nonnegative integer n the number $2n - 5$.

   (b) Assign to each real number x the number 7.

   (c) Assign to the number 10 the real number y.

   (d) Assign to each pair of distinct points in the plane the distance between them.

   (e) $y = -3$ (for all x)

   (f) $x = 4$ (for all y and z)

   (g) $x + y = 2$

   (h) $y = 2x^2 + 3$

   (i) $y^2 - 4 = x$

   (j) $y < 2x - 1$

   (k) $f(x) = -\sqrt{16 - x^2}$

   (ℓ) $x^2 + y^2 = 16$

2. Sketch the graphs of equations (e) - (ℓ) of Number 1.

3. A function f is completely defined by the table:

x	0	1	2	3	4
$f(x)$	-3	1	5	9	13

   (a) Describe the domain and range of f.

   (b) Write an equation with suitably restricted domain that defines f.

4. If $f : x \longrightarrow x^2 + 3x - 4$, find

   (a) $f(0)$

   (b) $f(2)$

   (c) $f(-1)$

   (d) $f(\sqrt{3})$

   (e) $f(2 - \sqrt{2})$

   (f) $f\big(f(1)\big)$ (Hint: This is the value of f at $f(1)$.)

5. If g is a function defined by $g(x) = \dfrac{2x}{\sqrt{5 - x^2}}$, find, if possible,

(a) $g(0)$

(b) $g(1)$

(c) $g(-1)$

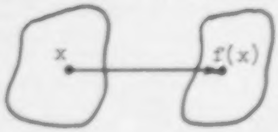
(d) $g(2)$

(e) $g(-3)$

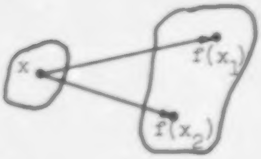
(f) $g(\sqrt{5})$

6. Which of the following mappings represent functions?

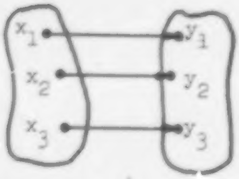
(a)



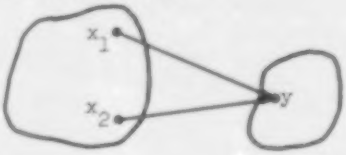
(b)



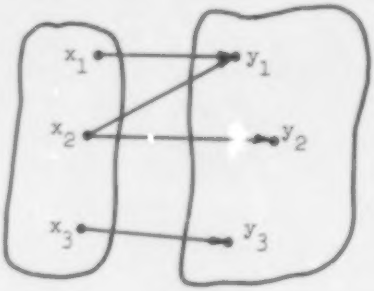
(c)



(d)



(e)



7. Given the functions $f : x \longrightarrow x$ and $g : x \longrightarrow \dfrac{x^2}{x}$. If x is a real number, are f and g the same function? Why or why not?

8. Given the functions $f : x \longrightarrow x + 2$ and $g : x \longrightarrow \dfrac{x^2 - 4}{x - 2}$. If x is real, are f and g the same function? Why or why not?

9. What number or numbers have the image 10 under the following mappings?

(a) $f : x \longrightarrow 2x$

(b) $g : x \longrightarrow x^2$

(c) $h : x \longrightarrow \sqrt{x^2 + 36}$

(d) $\alpha : x \longrightarrow |x - 4|$

(e) $\emptyset : x \longrightarrow [x]$

10. Which of the following statements are always true for any function f, assuming that x_1 and x_2 are in the domain of f?

(a) If $x_1 = x_2$, then $f(x_1) = f(x_2)$.

(b) If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

(c) If $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(d) If $f(x_1) \neq f(x_2)$, then $x_1 \neq x_2$.

Insert page 355

11. If $f(x) = |x|$, which of the following statements are true for all real numbers x and t?

    (a) f is an odd function.

    (b) $f(x^2) = f(x)^2$

    (c) $f(x - t) \leq f(x) - f(t)$

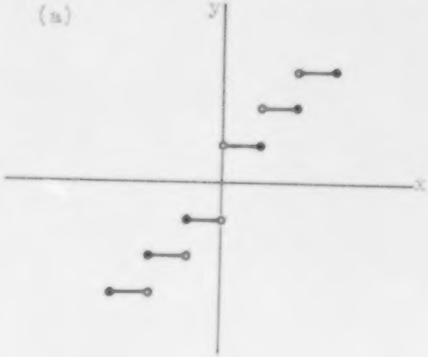
    (d) $f(x + t) \leq f(x) + f(t)$

12. Which of the following functions are even, which are odd, and which are neither even nor odd?

(a) $f : x \longrightarrow 3x$	(e) $f : x \longrightarrow x^3 + 4$
(b) $f : x \longrightarrow -2x^2 + 5$	(f) $f : x \longrightarrow x^3 - 2x$
(c) $f : x \longrightarrow x^2 - 4x + 4$	(g) $f : x \longrightarrow 2^{1/x}$
(d) $f : x \longrightarrow -2x + 1$	(h) $f : x \longrightarrow 2^{1/x^2}$

13. Which of the following graphs could represent functions?

(a)



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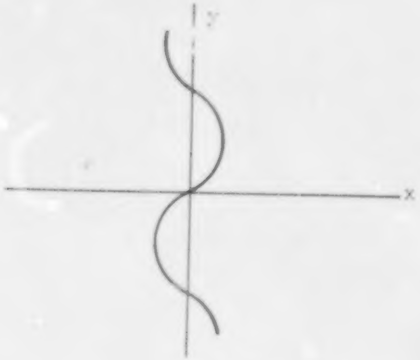
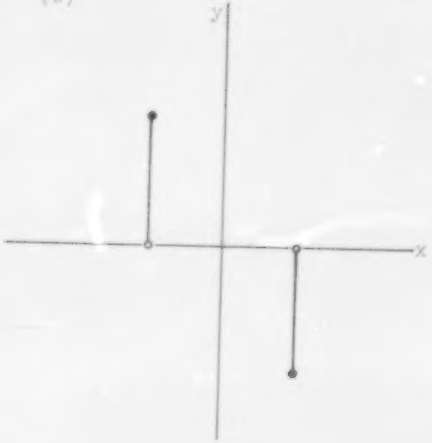
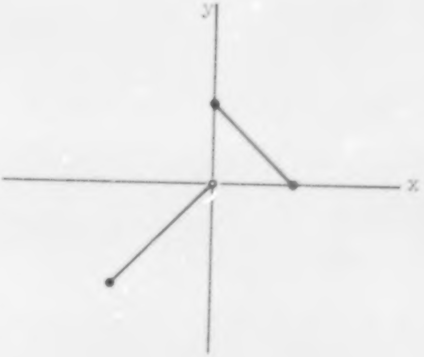


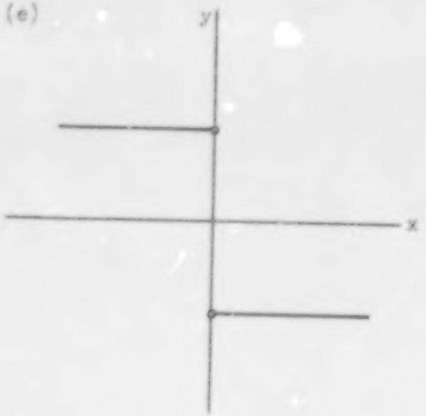
(b)



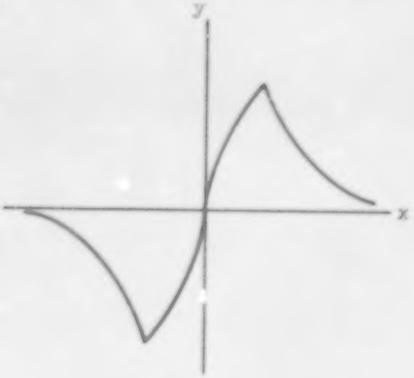
(d)



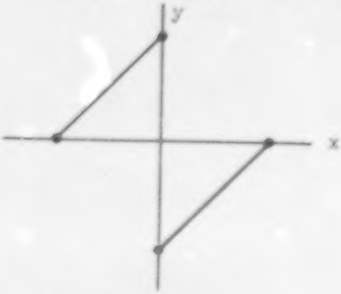
(e)



(f)



14. Suppose that $f : x \longrightarrow f(x)$ is the function whose graph is shown. Sketch the graphs of



(a) $g : x \longrightarrow -f(x)$

(b) $g : x \longrightarrow f(-x)$

(c) $g : x \longrightarrow |f(x)|$

(d) $g : x \longrightarrow f(|x|)$

15. A function f is defined by

$$f(x) = \begin{cases} \dfrac{x}{|x|} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Identify this function and sketch its graph.

16. Sketch the graph of each function, specifying its domain and range.

(a) $f : x \longrightarrow \sqrt{x^2}$

(b) $f : x \longrightarrow -|x|$

(c) $f : x \longrightarrow |1 - x|$

(d) $f : x \longrightarrow 1 - |x|$

(e) $f : x \longrightarrow |x| - x$

(f) $f : x \longrightarrow |x| + |x - 1|$

(Hint: Consider separately the three possibilities: $x < 0$, $0 \leq x \leq 1$, and $x > 1$.)

(g) $f : x \longrightarrow \operatorname{sgn} x$

(h) $f : x \longrightarrow [-x]$

(i) $f : x \longrightarrow \dfrac{[x]}{x}$

(j) $f : x \longrightarrow x[x]$

(k) $f : x \longrightarrow |1 - x^2|$

(ℓ) $f : x \longrightarrow |x^2 - 2x - 3|$

Sketch the graphs of the functions in Exercises 17 to 19. For those functions which are periodic, indicate their periods. Indicate those functions which are even or odd.

17. (a) $f : x \longrightarrow x - \left[x - \frac{1}{2}\right]$
    (b) $f : x \longrightarrow 2x^2 - [2x^2]$
    (c) $f : x \longrightarrow 2x^2 - 2[x^2]$
    (d) $f : x \longrightarrow 2x^2 - 2[x]^2$

18. (a) $f : x \longrightarrow ax - [ax], \quad a > 0$
    (b) $f : x \longrightarrow 5x - [2x] - [3x]$
    (c) $f : x \longrightarrow x(\sqrt{2} + 1) - [x\sqrt{2}] - [x]$

19. (a) $f : x \longrightarrow \frac{1 + \operatorname{sgn} x}{2}$. This function is also called the Heaviside unit function and is designated by $f : x \longrightarrow H(x)$.
    (b) $f : x \longrightarrow H(x) + H(x - 2)$
    (c) $f : x \longrightarrow H(x) \cdot H(x - 2)$
    (d) $f : x \longrightarrow (x - 2)^2 \cdot H(x)$
    (e) $f : x \longrightarrow H(x) + H(x - 2) + H(x - 4)$
    (f) $f : x \longrightarrow H(x^2 - 2)$
    (g) $f : x \longrightarrow (\operatorname{sgn} x)(x - 1)^2 + [\operatorname{sgn}(x - 1)]x^2$

20. If f and g are periodic functions of periods m and n, respectively (m, n integers), show that $f + g$ and $f \cdot g$ are also periodic. Give examples to show that the period of $f + g$ can either be greater or less than both of m and n. Repeat the same for the product $f \cdot g$.

21. (a) Can a function be both even and odd?
    (b) What can you say about the evenness or oddness of the product of:
        (1) an even function by an even function?
        (2) an even function by an odd function?
        (3) an odd function by an odd function?
    (c) Show that every function whose domain contains $-x$ whenever it contains x can be expressed as the sum of an even function plus an odd function.

22. Find functions $f(x)$ satisfying

$$f(x) \cdot f(-x) = 1 \quad \text{(called a functional equation).}$$

Suggestion: Use 21(c).

23. Prove that no periodic function other than a constant can be a rational function. (Note: A rational function is the ratio of two polynomial functions.)

A1-2. Composite Functions

Given two functions f and g with domains whose intersection is nonempty, we can construct new functions by using any of the elementary rational operations--addition, subtraction, multiplication, division--on the given functions. Thus, the sum of f and g is defined to be the function

$$f + g : x \longrightarrow f(x) + g(x)$$

which has for domain those elements contained in the intersection of the domains of f and g. Similarly there are definitions for the difference, product, and quotient of two functions; there is, in fact, a whole algebra of functions, just as there is the familiar algebra of real numbers.

In this algebra of functions there is one operation that has no counterpart in the algebra of numbers: the operation of composition. This operation is best explained by examples.

Let

$$g : x \longrightarrow 2x + 1$$

and

$$f : x \longrightarrow x^2.$$

We observe that

$$g(1) = 3 \text{ and } f(3) = 9,$$
$$g(2) = 5 \text{ and } f(5) = 25,$$

and, in general, the value of f at $g(x)$ is

$$f\big(g(x)\big) = f(2x + 1) = (2x + 1)^2.$$

We have constructed a new function which maps x onto the square of $(2x + 1)$. This function, defined by the mapping $x \longrightarrow f\big(g(x)\big)$ and denoted by fg, is called a composite of f and g. Hereafter we shall usually represent the value of the function fg by $fg(x)$ rather than $f\big(g(x)\big)$. Either symbol means the value of f at $g(x)$.*

---

[*] The symbol fg denoting the composit of the functions f and g must not be confused with the product of the functions. In this text we distinguish the latter by use of the dot for multiplication; i.e., $f \cdot g$.

An immediate question arises as to the order in which two functions are composed: is the composition of functions a commutative operation; i.e., in general, are $gf(x)$ and $fg(x)$ equal? In the example above we have seen that $fg(2) = f(5) = 25$, and we calculate $gf(2)$:

$$gf(2) = g(4) = 5 \neq fg(2).$$

This counterexample is sufficient to prove that in general $gf(x) \neq fg(x)$. The operation of composition applied to two functions f and g generally produces two different composite functions fg and gf, depending upon the order in which they are composed.

A word of caution must be injected at this point. The number $fg(x)$ is defined only if x is in the domain of g and $g(x)$ is in the domain of f. For example, if

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = 3x - 9,$$

then

$$fg(x) = f(3x - 9) = \sqrt{3x - 9},$$

and the domain of fg is the set of real numbers x for which $3x - 9$ is nonnegative; hence the domain is the set of all $x \geq 3$.

For the other composition of the same functions f and g, we have

$$gf(x) = g(\sqrt{x}) = 3\sqrt{x} - 9$$

which is defined for all nonnegative real numbers x.

We define composition of functions formally.

> The composite fg of two functions f and g is the function
>
> $$fg : x \longrightarrow fg(x) = f\big(g(x)\big).$$
>
> The domain of fg is the set of all elements x in the domain of g for which $g(x)$ is in the domain of f. The operation of forming a composite of two functions is called composition.

The definition may be extended to the composition of three or more functions. Thus, if f, g, and h are functions, one composite is

$$fgh : x \longrightarrow fgh(x) = f\Big(g\big(h(x)\big)\Big).$$

In order to evaluate $fgh(x)$, we first find $h(x)$, then the value of g at $h(x)$, and finally the value of f at $gh(x)$.

Exercises A1-2

1. Given that $f : x \longrightarrow x - 2$ and $g : x \longrightarrow x^2 + 1$ for all real x, find

    (a) $f(2) + g(2)$. (e) $f(x) + g(x)$.

    (b) $f(2) \cdot g(2)$. (f) $f(x) \cdot g(x)$.

    (c) $fg(2)$. (g) $fg(x)$.

    (d) $gf(2)$. (h) $gf(x)$.

2. If $f(x) = 3x + 2$ and $g(x) = 5$, find

    (a) $fg(x)$.

    (b) $gf(x)$.

3. If $f(x) = 2x + 1$ and $g(x) = x^2$, find

    (a) $fg(x)$ and $gf(x)$.

    (b) For what values of x, if any, are $fg(x)$ and $gf(x)$ equal?

4. For each pair of functions f and g, find the composite functions fg and gf and specify the domain (and range, if possible) of each.

    (a) $f : x \longrightarrow \frac{1}{x}$, $g : x \longrightarrow 2x - 6$

    (b) $f : x \longrightarrow \frac{1}{x}$, $g : x \longrightarrow x^2 - 4$

    (c) $f : x \longrightarrow \frac{1}{x}$, $g : x \longrightarrow \sqrt{x}$

    (d) $f : x \longrightarrow x^2$, $g : x \longrightarrow \sqrt{x}$

    (e) $f : x \longrightarrow x^2$, $g : x \longrightarrow \sqrt{4 - x}$

    (f) $f : x \longrightarrow -x^2 - 1$, $g : x \longrightarrow \sqrt{x}$

5. Given that $f(x) = x^2 + 3$ and $g(x) = \sqrt{x + 2}$, solve the equation

$$fg(x) = gf(x).$$

6. Solve problem 5 taking $g(x) = \sqrt{x - 2}$.

7. Describe functions f and g such that gf will equal

    (a) $3(x + 2) - 4$. (d) $\sqrt{x^2 - 4}$.

    (b) $(2x - 5)^3$. (e) $(x^4)^2$.

    (c) $\frac{3}{2x - 5}$.

8. For each pair of functions f and g find the composite functions fg and gf and specify the domain (and range, if possible) of each. Also, sketch the graph of each, and give the period (fundamental) of those which are periodic.

   (a) $f : x \longrightarrow |x|$, $g : x \longrightarrow \operatorname{sgn}(x - 2)$

   (b) $f : x \longrightarrow |x|$, $g : x \longrightarrow 2 \operatorname{sgn}(x - 2) - 1$

9. What can you say about the evenness or oddness of the composite of

   (a) an even function of an even function?

   (b) an even function of an odd function?

   (c) an odd function of an odd function?

   (d) an odd function of an even function?

10. If the function f is periodic, what can you say about the periodic character of the composite functions fg and gf assuming these exist and g is an arbitrary function (not periodic)? Illustrate by examples.

11. If the functions f and g are each periodic, then the composite functions fg and gf (assumed to exist) are also periodic. Can the period of either one be less than that of both f and g?

12. A sequence $a_0, a_1, a_2, \ldots, a_n, \ldots,$ is defined by the equation

$$a_{n+1} = f(a_n), \; n = 0, 1, 2, 3, \ldots,$$

where f is a given function and a_0 is a given number. If $a_0 = 0$ and $f : x \longrightarrow \sqrt{2 + x}$, then

$$a_1 = f(a_0) = \sqrt{2}$$

$$a_2 = f(a_1) = ff(a_0) = \sqrt{2 + \sqrt{2}}$$

$$a_3 = f(a_2) = ff(a_1) = fff(a_0) = \sqrt{2 + \sqrt{2 + \sqrt{2}}}.$$

Show that for any n,

(a) $a_n < 2$.

(b) $a_n > 2 - \frac{1}{2^{n-1}}$, $n > 0$.

13. If $a_{n+1} = f(a_n)$, $n = 0, 1, 2, \ldots$, $a_0 = \mu$, find a_n as a function of μ and n, for the following functions f:

(a) $f : x \longrightarrow a + bx$.

(b) $f : x \longrightarrow x^m$.

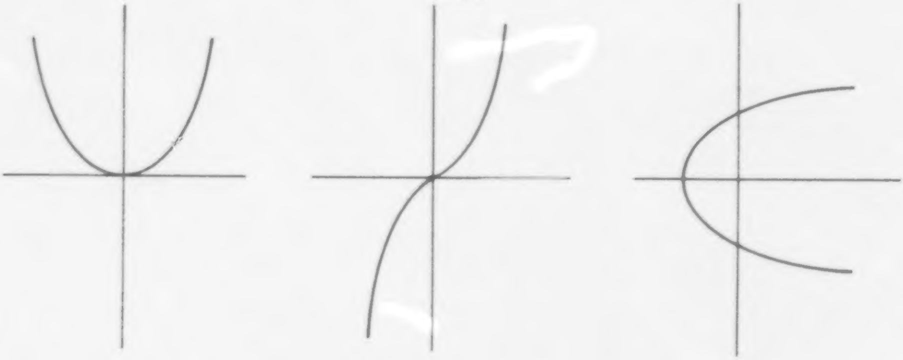
(c) $f : x \longrightarrow \sqrt{|x|}$.

(d) $f : x \longrightarrow \sqrt{1 - x^2}$.

(e) $f : x \longrightarrow (1 - x)^{-1}$.

A1-3. Inverse Functions

Recall the vertical line test for the graph of a function (Section A1-1): if every line which is parallel to the y-axis intersects a graph in at most one point, then the graph is that of a function. Thus in Figure A1-3a, (i) and (ii) illustrate graphs of functions, (iii) is the graph of a relation that is not a function.



(i) (ii) (iii)

Figure A1-3a

This figure also illustrates an important distinction between two classes of functions: for graph (i) there is at least one line parallel to the x-axis which intersects the graph in more than one point; this is not the case for graph (ii). The latter is typical of a class of functions called one-to-one functions: each element in the domain is mapped into one and only one image in the range, and each element in the range corresponds to one and only one preimage in the domain. In other words, a function of this kind establishes a one-to-one correspondence between the domain and the range of the function.

> A function f is one-to-one if whenever $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Note the distinction between the definition of function and this definition. The former states that any function f has the property that if $x_1 = x_2$, then $f(x_1) = f(x_2)$, whereas the latter states that a one-to-one function f is such that $f(x_1) = f(x_2)$ if and only if $x_1 = x_2$.

The class of one-to-one functions is important because for each member of this class we can specify a function that, in a loose way of speaking, undoes the work of the given function. Thus, for example, if f is the function which maps each real number onto its double, then there is a function g, called the inverse of f, which reverses this mapping and takes each real number onto its half: $f : x \longrightarrow 2x$; $g : y \longrightarrow \frac{1}{2}y$.

> If a function $f : x \longrightarrow f(x)$ is one-to-one, then the function $g : f(x) \longrightarrow x$, whose domain is the range of f, is called the inverse of f.

The functions f and g represent the same association but considered from opposite directions; the domain of g is the range of f and the range of g is the domain of f. Furthermore, g is itself one-to-one and its inverse is f.

It is instructive to look at the composites of two functions f and g inverse to one another. If f maps x into y, then g maps y back into x; in other words, if $y = f(x)$, then $x = g(y)$. Hence,

$$gf(x) = g(y) = x, \quad \text{for all } x \text{ in the domain of } f,$$

and

$$fg(y) = f(x) = y, \quad \text{for all } y \text{ in the range of } f.$$

Observe that the restriction of the domain of g to coincide with the range of f is part of the definition of the inverse.

Example A1-3a. Consider the one-to-one function $f : x \longrightarrow 2x - 3$; what is its inverse? Here f is described by the instruction, "Take a number, double it, and then subtract 3." In order to reverse this procedure, we must add 3 and then divide by 2. This suggests that the inverse of f is the function $g : x \longrightarrow \frac{x + 3}{2}$. To prove this fact, we must show that g satisfies the definition of inverse; i.e., show that g maps $f(x)$ into x for all x in the domain of f. By substitution,

$$gf(x) = g(2x - 3) = \frac{(2x - 3) + 3}{2} = x;$$

g is the inverse of f. Furthermore, in the opposite direction,

$$fg(x) = f(\frac{x + 3}{2}) = 2(\frac{x + 3}{2}) - 3 = x$$

for all x in the domain of g. Hence, f is the inverse of the function g, as expected.

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The graph of the inverse g of a function f is easily found from the graph of f. If f maps a into b, then g maps b into a. It follows that the point (a,b) is on the graph of f, if and only if (b,a) is on the graph of g. Figure A1-3b shows three points (1,-3), (2,1), and (4,2) on the graph of a function f, and their corresponding points, obtained by interchange of coordinates, on the graph of g.

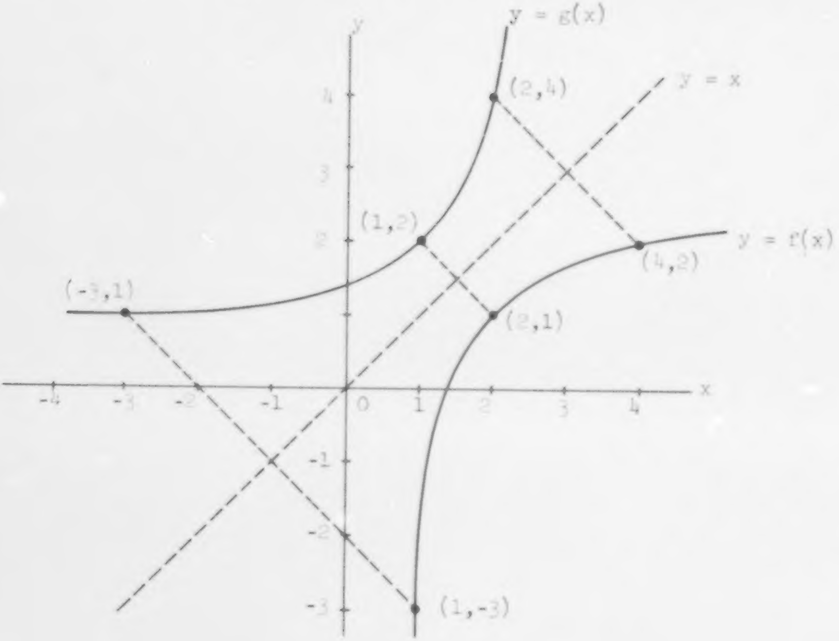


Figure A1-3b

From this figure we see that the points (a,b) and (b,a) are symmetric with respect to the line $y = x$; that is, the line segment determined by these two points is perpendicular to, and bisected by, the line $y = x$. We call (b,a) the reflection of (a,b) in the line $y = x$.

Example A1-3b. Consider the functions $f : x \longrightarrow \sqrt{x + 2}$, $x \geq -2$,, and $g : x \longrightarrow x^2 - 2$. The function f is one-to-one; g is not and, hence, cannot be the inverse of f as it stands. This can be restricted by restricting the domain of g to $x \geq 0$; i.e., the inverse of f is $g : x \longrightarrow x^2 - 2$, $x \geq 0$ (Figure A1-3c). The composite functions verify that f and g are inverse to one another:

$$fg : x \longrightarrow fg(x) = \sqrt{(x^2 - 2) + 2} = x, \quad x \geq 0;$$

$$gf : x \longrightarrow gf(x) = (\sqrt{x + 2})^2 - 2 = x, \quad x \geq -2.$$

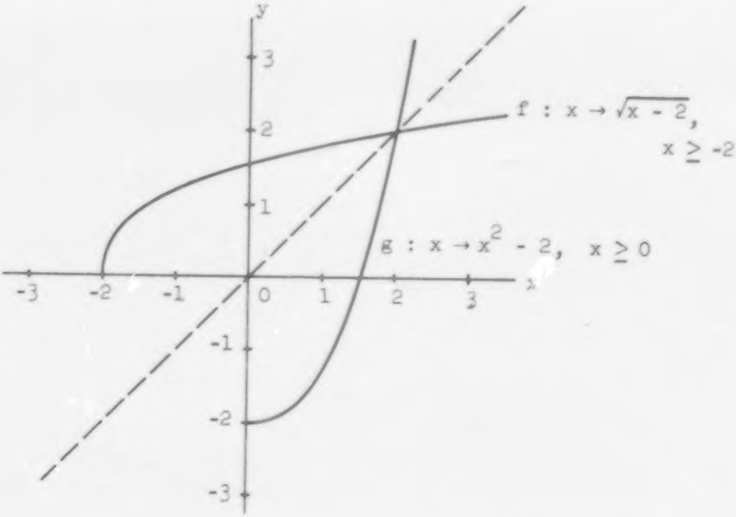


Figure A1-3c

The relationship between the coordinates of a point (a,b) and the coordinates of its reflection (b,a) in the line $y = x$ suggests a formal method for obtaining an equation of the inverse of a given function assuming that the inverse exists.

Example A1-3c. Consider the function

$$f : x \longrightarrow y = 3x + 5 \quad \text{for all real } x.$$

If we interchange x and y in the equation

(1) $$y = 3x + 5,$$

we obtain

(2) $$x = 3y + 5.$$

For every pair of numbers (a,b) in the solution set of (1), a pair (b,a) is in the solution set of (2). Hence, (2) is an equation defining implicitly the inverse of the given function f. In order to obtain the explicit form, we solve (2) for y in terms of x and obtain

$$y = \frac{x - 5}{3}.$$

The inverse of f is, therefore,

$$g : x \longrightarrow g(x) = \frac{x - 5}{3} \qquad \text{for all real } x.$$

You should verify the fact that $gf(a) = a$ for any a in the domain of f, and that $fg(b) = b$ for any b in the domain of g (range of f).

Example A1-3d. If the given equation defines a quadratic function, the problem of finding an inverse is more complicated. In the first place, the given function must be restricted to a domain which gives a one-to-one function; in the second place, the technical details of interchanging the variables x and y in the given equation and then solving for y are more involved.

Consider the function

$$f : x \longrightarrow x^2 + 2x + 3$$

whose graph is a parabola with vertex at $(-1,2)$ and opening upward. If, for example, we restrict f to the domain $\{x : x \geq -1\}$, then we have a function f_1 which is one-to-one and hence has an inverse g_1. The range of f_1 is $\{y : y = f_1(x) \geq 2\}$, and this will be the domain of g_1.

We proceed to find a formula defining g_1. We are given

$$y = x^2 + 2x + 3,$$

and we interchange the variables to obtain

$$x = y^2 + 2y + 3.$$

We now solve for y in the quadratic equation

$$y^2 + 2y + (3 - x) = 0,$$

obtaining

$$y = -1 + \sqrt{x - 2} \quad \text{or} \quad y = -1 - \sqrt{x - 2}.$$

Which of these formulas defines the function g_1? Since y here represents any element in the range of the inverse function, and since the range must be the same set of numbers as the domain of f_1, we see that $y \geq -1$ is required. Hence

$$y = -1 + \sqrt{x - 2}$$

defines the inverse function

$$g_1 : x \longrightarrow -1 + \sqrt{x - 2}$$

whose domain is $\{x : x \geq 2\}$. (Note, again, that this is the range of f_1.)

It is helpful to sketch the graphs of the two inverse functions in order to see more clearly the relationships between their domains and ranges. (See Figure A1-3d.) In fact, if you graph the original function f, you may see more clearly how its domain may be restricted in infinitely many ways to give as many different one-to-one functions, each of which has a unique inverse function.

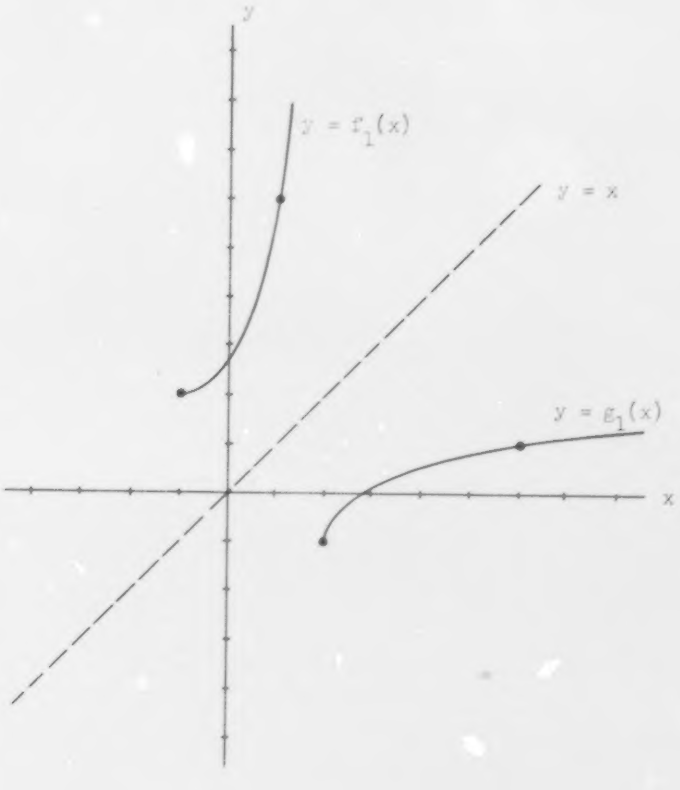


Figure A1-3d

Exercises A1-3

1. What is the reflection of the line $y = f(x) = 3x$ in the line $y = x$? Write an equation defining the inverse of f.

2. Which points are their own reflections in the line $y = x$? What is the graph of all such points?

3. (a) Find the slope of the segment from (a,b) to (b,a), and prove that the segment is perpendicular to the line $y = x$.

   (b) Prove that the segment from (a,b) to (b,a) is bisected by the line $y = x$.

4. What is the reflection of $(1,1)$ in the line

   (a) $x = 0$? (d) $y = 2$?

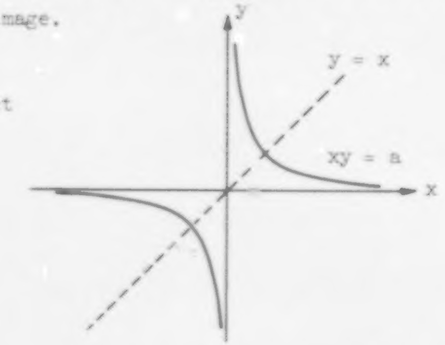
   (b) $y = 0$? (e) $x = -3$?

   (c) $y = -x$?

5. Describe any function or functions you can think of which are their own inverses.

6. An equation or an expression (phrase) is said to be symmetric in x and y if the equations or the expressions remain unaltered by interchanging x and y; e.g., $x^2 + y^2 = 0$, $x^3 + y^3 - 3xy$, $|x - y| = |x + y|$, $x - xy + y$. It follows that graphs of symmetric equations are symmetric about the $y = x$ line. Geometrically, we can consider the line $y = x$ behaving as a mirror, i.e., for any portion of the graph there must also be a portion which is the mirror image.

   The equation $x^4 + y^4 = a^4$ is obviously symmetric with respect to the line $y = x$. What other axes of symmetry (mirror type) does it have?



7. The expression

$$a + b + |a - b| + 2c + \Big|a + b + |a - b| - 2c\Big|$$

is obviously symmetric in a and b. Show that it is also symmetric in a and c.

Hint: Show six cases (i) $a < b < c$, (ii) $a < c < b$, (iii) $b < a < c$, etc.

8. Find the inverse of each function.

   (a) $f : x \longrightarrow 3x + 6$

   (b) $f : x \longrightarrow x^3 - 5$

   (c) $f : x \longrightarrow \frac{2}{x} - 3$

9. Which of the following functions have inverses? Describe each inverse by means of a graph or equation and give its domain and range.

   (a) $f : x \longrightarrow x^2$

   (b) $f : x \longrightarrow \sqrt{x}$

   (c) $f : x \longrightarrow |x|$

   (d) $f : x \longrightarrow [x]$

   (e) $f : x \longrightarrow x|x|$

   (f) $f : x \longrightarrow \operatorname{sgn} x$

10. As we have seen, $f : x \longrightarrow x^2$ for all real x does not have an inverse. Do the following:

    (a) Sketch graphs of $f_1 : x \longrightarrow x^2$ for $x \geq 0$ and $f_2 : x \longrightarrow x^2$ for $x \leq 0$, and determine the inverses of f_1 and f_2.

    (b) What relationship exists among the domains of f, f_1, and f_2? (f_1 is called the restriction of f to the domain $\{x : x \geq 0\}$, and f_2 is similarly the restriction of f to the domain $\{x : x \leq 0\}$.)

11. (a) Sketch a graph of $f : x \longrightarrow \sqrt{4 - x^2}$ and show that f does not have an inverse.

    (b) Divide the domain of f into two parts such that the restriction of f to either part has an inverse.

    (c) Write an equation defining each inverse of part (b) and sketch the graphs.

12. Do Problem 11 for $f : x \longrightarrow x^2 - 4x$.

13. Given that $f(x) = 3x - 2$ and $g(x) = -2x + k$,, find k such that $fg(x) = gf(x)$. For this value of k, are f and g inverse to one another? Give reasons for your answers.

14. Show that $f : x \longrightarrow x^2 - 4x + 5$ for $x \geq 2$ and $g : x \longrightarrow 2 + \sqrt{x - 1}$ for $x \geq 1$ are inverse to one another by showing that $fg(y) = y$ for all y in the domain of g, and that $gf(x) = x$ for all x in the domain of f.

15. If $f(x) = (2x^3 + 1)^7$, find at least two functions g such that $fg(x) = gf(x)$.

A1-4. Monotone Functions

If we examine the behavior, for x increasing, of the functions $f : x \longrightarrow \sqrt{x}$ and $g : x \longrightarrow \sin x$, we note that the values of f increase as x increases, while the values of g are sometimes increasing and sometimes decreasing. Geometrically this means that the graph of f is continually rising as we survey it from left to right (the direction of increasing x), whereas the graph of g, like a wave, is now rising, now falling. The graph of a function may also contain horizontal portions (parallel to the x-axis), where the values of the function remain constant on an interval. A function such as $x \longrightarrow [x]$ illustrates this, and also points up the fact that the graph of such a function need not be continuous.

Example A1-4a. The function h, defined by

$$h(x) = \begin{cases} -x^2 , & 0 \le x \le 1; \\ -1 , & 1 \le x \le 2; \\ -\frac{x^3}{8} , & 2 \le x, \end{cases}$$

has the graph shown in Figure A1-4a.

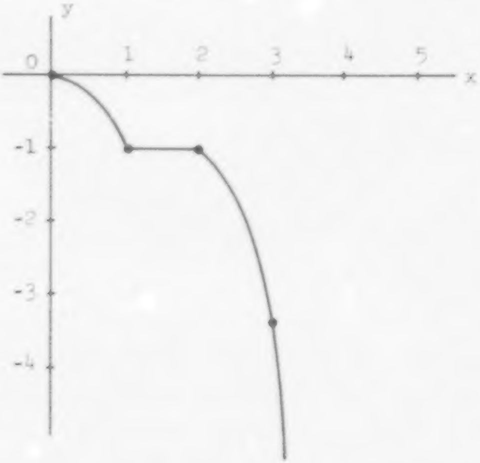


Figure A1-4a

It is easy to see that the function decreases as x increases except on the interval $[1,2]$ on which it remains constant.

Taken as a class, the increasing, decreasing, strictly increasing, and strictly decreasing functions are called monotone (compare with monotonous) because the changes in the values of the functions as x increases are always in one direction.

> Let f be a function defined on an interval I and let $y_1 = f(x_1)$, $y_2 = f(x_2)$ for x_1, x_2 in I. If, for each pair of numbers x_1 and x_2 in I, with $x_1 < x_2$, the corresponding values of y satisfy the inequality
>
> (1) $y_1 < y_2$, then f is a strictly increasing function;
>
> (2) $y_1 > y_2$, then f is a strictly decreasing function;
>
> (3) $y_1 \leq y_2$, then f is an increasing function;
>
> (4) $y_1 \geq y_2$, then f is a decreasing function.*

Briefly, this definition states that a function which preserves order relations is increasing; a function which reverses order relations is decreasing. Note particularly that a strictly increasing function is a special case of an increasing function; similarly, a strictly decreasing function is a special case of a decreasing function.

> A function which is either increasing or decreasing is called monotone. A function which is either strictly increasing or strictly decreasing is called strictly monotone.

For example, the function h of Example A1-4a is monotone over its entire domain and strictly monotone on the closed interval $0 \leq x \leq 1$ as well as on the interval $x \geq 2$.

The graph of a strictly monotone function suggests that the function must be one-to-one, hence must have an inverse.

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*In some texts the term "nondecreasing" is used instead of "increasing"; "nonincreasing" is used instead of "decreasing." In Volume 1 of this book we usually drop the phrase "strictly" from these definitions, using (1) or (3) as the definition of "increasing."

THEOREM A1-4. If a function is strictly monotone, then it has an inverse which is strictly monotone in the same sense.

Proof. We treat the case for f strictly increasing; the proof for f strictly decreasing is entirely similar. If $x_1 \neq x_2$, take $x_1 < x_2$, then $f(x_1) < f(x_2)$; that is, $f(x_1) \neq f(x_2)$. Hence, f is one-to-one, and f has an inverse

$$g : f(x) \longrightarrow x$$

defined for all values $f(x)$ in the range of f.

Finally, g is strictly increasing, for if y_1 and y_2 are in the domain of g and $y_1 < y_2$, then $y_1 = f(x_1)$, $y_2 = f(x_2)$ and x_1 must be less than x_2. (Why?) Therefore, $g(y_1) = x_1 < x_2 = g(y_2)$.

Example A1-4b. The function

$$f : x \longrightarrow x^n,$$

n a natural number, is strictly monotone (increasing) for all real $x \geq 0$. Hence, f has the inverse function

$$g : x^n \longrightarrow x, \qquad x \geq 0, \tag{1}$$

which is also an increasing function. For an arbitrary element y in the domain of g, we denote $g(y)$ by $\sqrt[n]{y}$; thus (1) may be rewritten

$$g : y \longrightarrow \sqrt[n]{y}, \qquad y \geq 0. \tag{2}$$

Comparing (1) and (2), we see that $\sqrt[n]{y}$ is the unique positive solution x of the equation $x^n = y$; we call $\sqrt[n]{y}$ the n-th root of y for all real $y \geq 0$.

If the natural number n is odd, then the function $f : x \longrightarrow x^n$ is strongly monotone for all real x, as is its inverse function. This means that every real number has a unique n-th root for n odd. For example, for n odd and a real, $\sqrt[n]{-a^n} = -a$.

If n is even, $f : x \longrightarrow x^n$ is decreasing for all real $x \leq 0$, and increasing for all real $x \geq 0$. If f_1 is the restriction of f to the domain $x \geq 0$ and f_2 is the restriction of f to $x \leq 0$, then each of these functions has an inverse, namely

$$g_1 : y \longrightarrow \sqrt[n]{y}$$

and

$$g_2 : y \longrightarrow -\sqrt[n]{y}$$

for n even and all real $y \geq 0$. For n even, the positive n-th root of a nonnegative real number is sometimes called its principal n-th root. The symbol $\sqrt[n]{y}$ always means the principal n-th root.

Exercises A1-4

1. Prove that $f : x \longrightarrow x^2$ for $x \geq 0$ is a strictly increasing function. (Hint: Let $x_1 > x_2 \geq 0$; then $x_1 - x_2 > 0$. From this show that $x_1^2 > x_2^2$.)

2. Which of the following functions are decreasing? increasing? strictly decreasing? strictly increasing? In each case the domain is the set of real numbers unless otherwise restricted.

   (a) $f_1 : x \longrightarrow c$, c a constant

   (b) $f_2 : x \longrightarrow x$

   (c) $f_3 : x \longrightarrow |x|$

   (d) $f_4 : x \longrightarrow [x]$

   (e) $f_5 : x \longrightarrow \operatorname{sgn} x$

   (f) $f_6 : x \longrightarrow -x^2, \quad x \leq 0$

   (g) $f_7 : x \longrightarrow -\sqrt{x}, \quad x \geq 0$

   (h) $f_8 : x \longrightarrow x|x|$

   (i) $f_9 : x \longrightarrow x + |x|$

   (j) $f_{10} : x \longrightarrow |x| + |x - 1|$

   (k) $f_{11} : x \longrightarrow |x - 1| + |x - 3|$

   (ℓ) $g_1 : x \longrightarrow f_3 f_4(x)$

   (m) $g_2 : x \longrightarrow f_4 f_3(x)$

3. For each function in Problem 2 which is not monotone, divide its domain into parts such that the restriction of f to any of these parts gives a monotone or strictly monotone function.

4. We are given that the function

f_1 is increasing,
f_2 is strictly increasing,
g_1 is decreasing,
g_2 is strictly decreasing,

in a common domain. What is the monotone character, if any, of the [illegible]ing functions:

(a)	$f_1 + f_2$.	(i)	$f_1 f_2$.
(b)	$f_2 + g_1$.	(j)	$f_2 f_1$.
(c)	$g_1 + g_2$.	(k)	$f_2 g_1$.
(d)	$g_2 + f_1$.	(ℓ)	$g_1 f_2$.
(e)	$f_1 \cdot f_2$.	(m)	$g_1 g_2$.
(f)	$f_2 \cdot g_1$.	(n)	$g_2 g_1$.
(g)	$g_1 \cdot g_2$.	(o)	$g_2 f_1$.
(h)	$g_2 \cdot f_1$.	(p)	$f_1 g_2$.

A1-5. Polar Coordinates

A fundamental link between the algebra of numbers and the geometry of points was forged by René Descartes (1596 - 1650) when he introduced the notion of representing a point in the plane by means of an ordered pair of numbers. Beautiful in its simplicity, this concept paved the way for the development of coordinate geometry and calculus. By introducing a pair of perpendicular number lines (coordinate axes), Descartes was able to assign to each point in the plane a unique pair of real numbers. We call these numbers the (rectangular) cartesian coordinates of the point; you have used them ever since you first began to represent equations by their graphs.

Other coordinate systems have been invented since Descartes' time because they are better adapted to treat some problems which are awkward to handle in cartesian coordinates. We consider here the polar coordinate system and a few examples of its use.

We suppose that we already have a rectangular (cartesian) coordinate system in the plane. We locate a point P in the plane by polar coordinates, an ordered pair of real numbers $(r,\theta)^*$ where $|r|$ is the length of the segment $\overline{OP}$ (sometimes called the radius vector) and θ is the direction angle made by $\overline{OP}$ with the positive x-axis (polar axis) (Figure A1-5a).

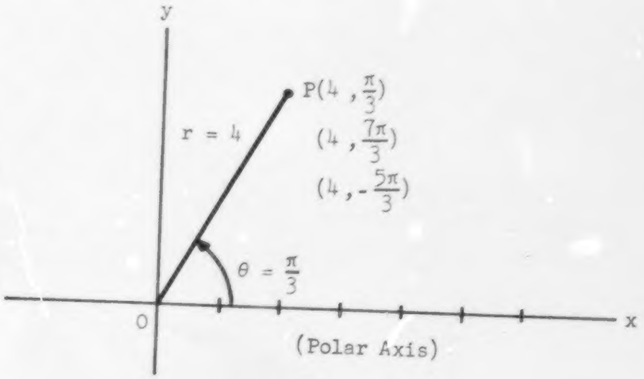

Figure A1-5a

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[*] r is sometimes called the radial coordinate and θ the polar angle or azimuth.

There are infinitely many such angles for each point P: if θ is one angle, then $\theta \pm 2n\pi$ $(n = 1, 2, 3, \ldots)$ are the others. Thus, a point may be identified by infinitely many pairs of polar coordinates. For example (Figure A1-5a), point P with polar coordinates $(4, \frac{\pi}{3})$, also has coordinates $(4, \frac{7\pi}{3})$, $(4, -\frac{5\pi}{3})$, and, in general, $(4, \frac{\pi}{3} + 2n\pi)$ for any integer n. The pole (origin) is a special case: to it we assign as polar coordinates any pair $(0, \theta)$, θ any real number.

When we assign polar coordinates to locate a point, it is customary to allow r also to be negative. For $r > 0$, the point $(-r, \theta)$ is located symmetrically to the point (r, θ) with respect to the origin (Figure A1-5b); it has coordinates $(r, \theta + \pi)$ also.

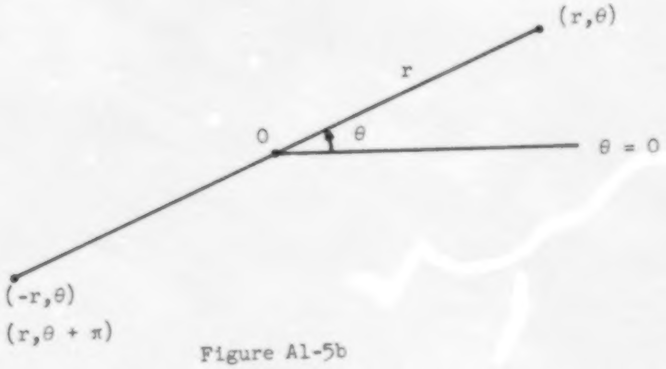


Figure A1-5b

In a cartesian coordinate system every point in the plane has a unique pair of coordinates (x, y). In a polar coordinate system, by contrast, this is not true; a given point in the plane does not have a unique representation (r, θ) in polar coordinates (see point P in Figure A1-5a). In both coordinate systems, however, a given pair of coordinates specifies a unique point in the plane.

A relation between x and y may be represented by a graph in a cartesian coordinate plane. A relation in r and θ may be represented by a graph in a polar coordinate system; a point lies on the graph if and only if it has at least one coordinate pair which satisfies the given relation. We discuss the graphs of a few functions defined by equations in polar coordinates.

The graph of the equation

$$r = c$$

contains all points (c,θ), θ any real number; it is a circle of radius $|c|$ having its center at the pole. The equation $r = -c$ describes the same circle.

The points for which

$$\theta = c$$

lie on the line passing through the pole which forms an angle of c radians with the polar axis; each point of the line has coordinates (r,c) for some real r. For r positive, the points form the ray in direction θ, for r negative, the ray has direction $\theta + \pi$. The line has infinitely many equations $\theta = c + n\pi$, n an integer.

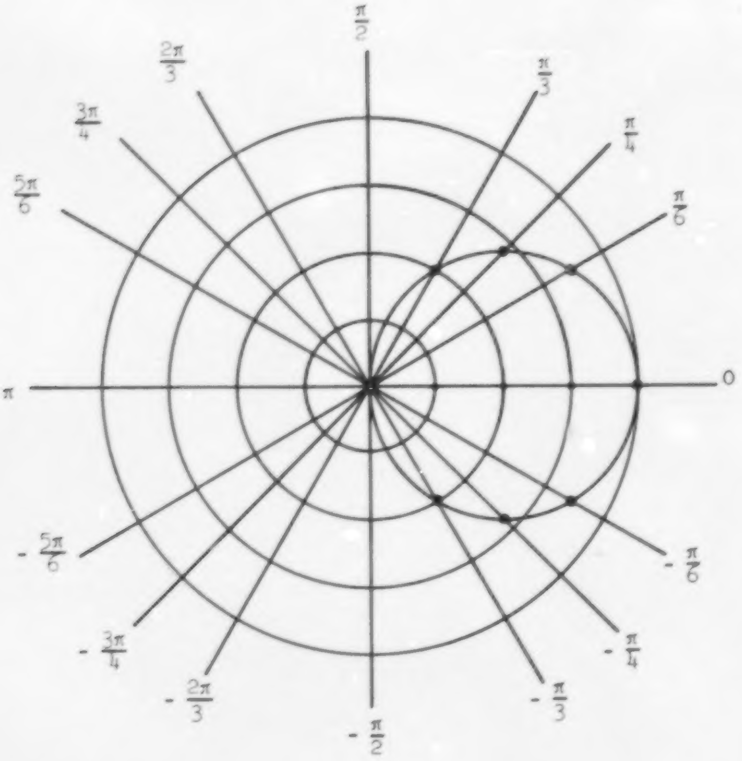
The circular functions of θ are especially conveniently represented in polar coordinates because the entire graph is traced out in one period. We shall illustrate a procedure for sketching a graph of such a function using polar coordinate graph paper. Note that the function specifies the graph; a function, however, cannot be recovered from its graph in polar coordinates.

Example A1-5a. Sketch a graph of the function defined by

$$r = 4 \cos \theta.$$

Since r is a function of θ, we consider values of θ and calculate the corresponding values of r. We know that the cosine increases from the value 0 at $\theta = -\frac{\pi}{2}$ to 1 at $\theta = 0$ and then decreases to 0 at $\theta = \frac{\pi}{2}$. Hence, in this interval, r increases from 0 to 4 and then decreases to 0. Since $\cos(\theta + \pi) = -\cos \theta$, the point $(4 \cos(\theta + \pi), \theta)$ is the same as $(-4 \cos \theta, \theta)$, and the curve for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is the entire graph.

To sketch the graph of the function, we calculate r for a few convenient values of θ $(-\frac{\pi}{3}, -\frac{\pi}{4}, \frac{\pi}{6}$, etc.), locate the corresponding points on polar coordinate paper, and sketch the graph (Figure A1-5c); it appears to be a circle and we shall presently verify that it is.



$\frac{\pi}{2}$
$\frac{2\pi}{3}$
$\frac{\pi}{3}$
$\frac{3\pi}{4}$
$\frac{\pi}{4}$
$\frac{5\pi}{6}$
$\frac{\pi}{6}$
π
0
$-\frac{5\pi}{6}$
$-\frac{\pi}{6}$
$-\frac{3\pi}{4}$
$-\frac{\pi}{4}$
$-\frac{2\pi}{3}$
$-\frac{\pi}{3}$
$-\frac{\pi}{2}$

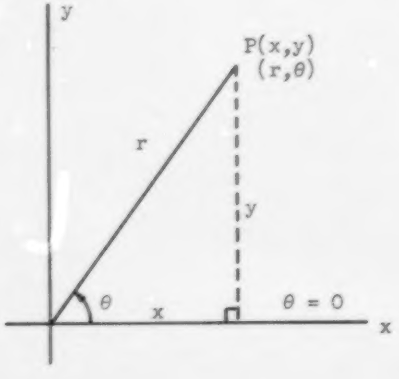
Figure A1-5c

Since each point P in the plane has both rectangular and polar coordinates (Figure A1-5d), for $r > 0$, we have from the trigonometric functions of angles

(1) $$x = r \cos \theta, \; y = r \sin \theta.$$

We leave it to you to verify that equations (1) hold for $r \leq 0$. Thus the rectangular and polar coordinates of each point in the plane are related by (1). It follows that

(2) $$x^2 + y^2 = r^2.$$



y
P(x,y)
(r,θ)
r
y
θ
x
θ = 0
x

Figure A1-5d

Now we re-examine the function defined by $r = 4 \cos \theta$ (Example A1-4a) and prove that its graph is a circle. We shall do so by transforming the given equation into an equation involving rectangular coordinates x and y. Now the given equation $r = 4 \cos \theta$ has the same graph as the equation

$$r^2 = 4r \cos \theta, \tag{3}$$

for if $r \neq 0$, we may divide both members of the equation by r to obtain the given equation; $r = 0$ corresponds to the fact that the pole is on both graphs. This may not be immediately obvious since only certain pairs of coordinates representing the pole will satisfy the equation $r = 4 \cos \theta$. For example, both $(0,0)$ and $(0, \frac{\pi}{2})$ represent the pole, yet only the latter of these pairs satisfies $r = 4 \cos \theta$.

We use (1) and (2) to obtain from (3) that

$$x^2 + y^2 = 4x$$

or

$$(x - 2)^2 + y^2 = 4,$$

an equivalent equation in rectangular coordinates. We recognize this as an equation of the circle with center at $(2,0)$ and radius 2, verifying the graph in Figure A1-5c.

Example A1-5b. Find an equation in polar coordinates of the curve whose equation in cartesian coordinates is $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

Applying Equations (1) and (2), we have

$$\begin{aligned} r^4 &= a^2r^2(\cos^2\theta - \sin^2\theta) \\ &= a^2r^2 \cos 2\theta. \end{aligned}$$

This is equivalent to

$$r^2 = 0 \text{ (the pole) and } r^2 = a^2 \cos 2\theta.$$

Since $r^2 = a^2 \cos 2\theta$ is satisfied by $(0, \frac{\pi}{4})$, a set of polar coordinates for the pole, we see that $r^2 = 0$ contributes no points not in the graph of $r^2 = a^2 \cos 2\theta$. Hence, the latter is an equation in polar form which is the equivalent of the given one. The graph of this equation is called the lemniscate of Bernoulli and is displayed in Figure A1-5e.*

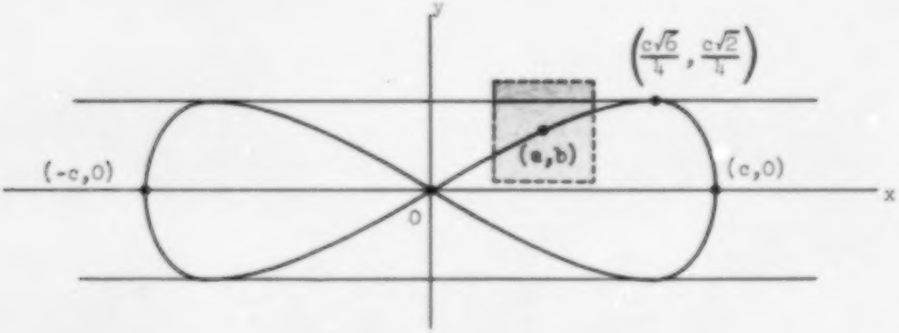


Figure A1-5e

We now develop an equation which, for suitable choice of a parameter, will represent either a parabola, an ellipse, or a hyperbola. For this purpose we need the definition of these curves (conic sections) in terms of focus, directrix, and eccentricity. Every conic section (other than the circle) may be defined to be the set (locus) of all points P such that the ratio of the distance between P and a fixed point F (the focus) to the distance between P and a fixed line l (the directrix) is a positive constant e, called the eccentricity of the conic section. If $e = 1$ the conic section is a parabola, if $0 < e < 1$ it is an ellipse, and if $e > 1$ it is a hyperbola.

In order to derive an equation in polar coordinates of a conic section, it is convenient to place the focus F at the pole (origin) and the directrix l perpendicular to the extension of the polar axis at distance $p > 0$ from the pole, as shown in Figure A1-5f. (Other orientations are possible; see Exercises A1-5, Nos. 8-10.) Point P is any point of the conic section.

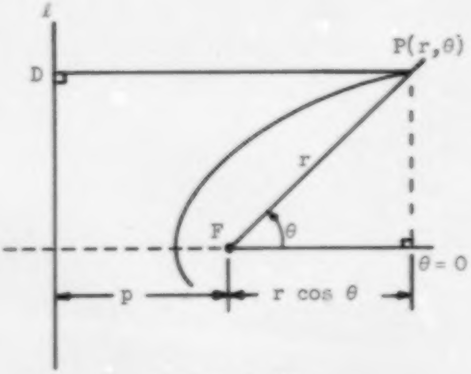


Figure A1-5f

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[*] This curve is defined as the set (locus) of points P such that the product of the distances of P from two fixed points is the square of half the distance between the two fixed points.

We let (r,θ) be any pair of polar coordinates of P for which $r > 0$; then $FP = r$ and $DP = p + r \cos \theta$ (Figure A1-5f). The definition of the conic sections requires that $\frac{FP}{DP} = e$ or $\frac{r}{p + r \cos \theta} = e$. Solving for r we obtain

$$r = \frac{ep}{1 - e \cos \theta} \tag{4}$$

which we take to be the standard form of the polar equation of conic sections having focus and directrix oriented as in Figure A1-5f. From Equation (4), if $e \leq 1$ (ellipse or parabola), then $r > 0$; if $e > 1$ (hyperbola), r may be negative and these values give us the branch of the hyperbola lying to the left of the directrix.

Example A1-5c. Describe and sketch the graph of the equation

$$r = \frac{16}{5 - 3 \cos \theta} .$$

We may put this equation in the standard form

$$r = \frac{\frac{16}{5}}{1 - \frac{3}{5} \cos \theta}$$

$$= \frac{\frac{3}{5} \cdot \frac{16}{3}}{1 - \frac{3}{5} \cos \theta} ,$$

from which $e = \frac{3}{5}$ and $p = \frac{16}{3}$. Since $e < 1$, the graph is an ellipse with focus F_1 at the pole and major axis on the polar axis. By giving θ the values 0 and π, we find the ends of the major axis to be $(8,0)$ and $(2,\pi)$. Thus the length of the major axis is 10, the center of the ellipse is the point $(3,0)$, and the other focus is the point $F_2(6,0)$. Since $p = \frac{16}{3}$ (the distance between a focus and corresponding directrix of the ellipse), the equation of the directrix ℓ_1 corresponding to the focus at the pole is $r \cos \theta = -\frac{16}{3}$ (see Exercises A1-5, No. 6a.), and the equation of the directrix ℓ_2 corresponding to $F_2(6,0)$ is $r \cos \theta = \frac{34}{3}$. When $\theta = \frac{\pi}{2}$, then $r = \frac{16}{5}$, and we have the point $(\frac{16}{5}, \frac{\pi}{2})$ at one end of the focal chord (latus rectum) through F_1. The other endpoint has polar coordinates $(\frac{16}{5}, \frac{3\pi}{2})$; these points help us to sketch the ellipse as shown in Figure A1-5g.

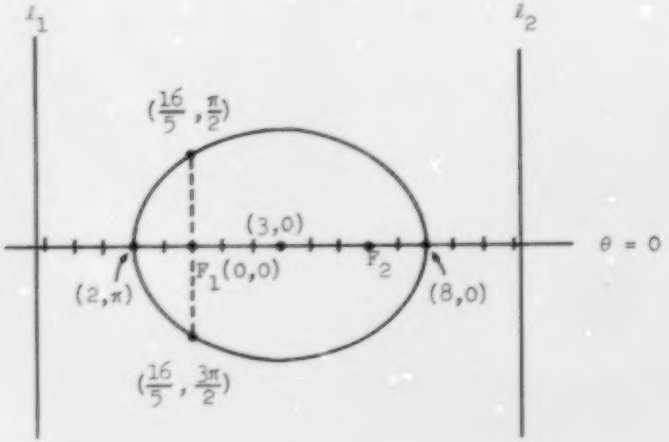
l_1
l_2
$(\frac{16}{5}, \frac{\pi}{2})$
$(3,0)$
$\theta = 0$
$F_1(0,0)$
F_2
$(2,\pi)$
$(8,0)$
$(\frac{16}{5}, \frac{3\pi}{2})$

Figure A1-5g

Exercises A1-5

1. Find all polar coordinates of each of the following points:

   (a) $(6, \frac{\pi}{4})$.

   (b) $(-6, \frac{\pi}{4})$.

   (c) $(6, -\frac{\pi}{4})$.

   (d) $(-6, -\frac{\pi}{4})$.

2. Find rectangular coordinates of the points in Exercise 1.

3. Find polar coordinates of each of the following points given in rectangular coordinates:

   (a) $(4, -4)$.

   (b) $(\frac{-3\sqrt{3}}{2}, \frac{3}{2})$.

   (c) $(-2, -2\sqrt{3})$.

   (d) $(0, -10)$.

   (e) $(-3, 0)$.

   (f) $(-3, 4)$.

   (g) $(-\sqrt{3}, 1)$.

   (h) $(\sqrt{2}, -\sqrt{2})$.

4. Given the cartesian coordinates (x,y) of a point, formulate unique polar coordinates (r,θ) for $0 \le \theta \le \pi$. (Hint: use $\arccos \frac{x}{r}$.)

5. Determine the polar coordinates of the three vertices of an equilateral triangle if a side of the triangle has length L, the centroid of the triangle coincides with the pole, and one angular coordinate of a vertex is θ_1 radians.

6. Find equations in polar coordinates of the following curves:

   (a) $x = c$, c a constant.

   (b) $y = c$, c a constant.

   (c) $ax + by = c$.

   (d) $x^2 + (y - k)^2 = k^2$.

   (e) $y^2 = 4ax$.

   (f) $x^2 - y^2 = a^2$.

7. Find equations in rectangular coordinates of the following curves:

   (a) $r = a$.

   (b) $r \sin \theta = -5$.

   (c) $r = 2a \sin \theta$.

   (d) $r = \frac{1}{1 - \cos \theta}$.

   (e) $r = 2 \tan \theta$.

8. Derive an equation in polar coordinates for conic sections with a focus at the pole and directrix perpendicular to the polar axis and p units to the right of the pole.

9. Repeat Number 8 if the directrix is parallel to the polar axis and p units above the focus at the pole.

10. Repeat Number 8 if the directrix is parallel to the polar axis and p units below the focus at the pole.

11. Discuss and sketch each of the following curves in polar coordinates. (See Example A1-5c and Nos. 8, 9, 10.)

    (a) $r = \frac{8}{1 - \cos\theta}$

    (b) $r = \frac{12}{1 - 3\cos\theta}$

    (c) $r = \frac{36}{5 - 4\sin\theta}$

    (d) $r = \frac{16}{5 + 3\sin\theta}$

    (e) $r \sin\theta = 1 - r$

12. Certain types of symmetry of curves in polar coordinates are readily detected. For example, a curve is symmetric about the pole if the equation is unchanged when r is replaced by $-r$. What kind of symmetry occurs if an equation is unchanged when

    (a) θ is replaced by $-\theta$?

    (b) θ is replaced by $\pi - \theta$?

    (c) r and θ are replaced by $-r$ and $-\theta$, respectively?

    (d) θ is replaced by $\pi + \theta$?

13. Without actually sketching the graphs, describe the symmetries of the graphs of the following equations:

    (a) $r^2 = 4\sin 2\theta$.

    (b) $r(1 - \cos\theta) = 10$.

    (c) $r = \cos^2 2\theta$.

14. Sketch the following curves in polar coordinates:

(a) $r = a\theta$.

(b) $r = a(1 - \cos\theta)$.

(c) $r = a \sin 2\theta$.

(d) $r = a^2 \sin^2\theta \cos^2\theta$.

(e) $r\theta = a$.

15. In each of the following, find all points of intersection of the given pairs of equations. (Recall that the polar representation of a point is not unique.)

(a) $r = 2 - 2\sin\theta$, $r = 2 - 2\cos\theta$

(b) $r = -2\sin 2\theta$, $r = 2\cos\theta$

(c) $r = 4(1 + \cos\theta)$, $r(1 - \cos\theta) = 3$

Appendix 2

POLYNOMIALS

A2-1. Significance of Polynomials

The importance of polynomials in applications to engineering and the natural sciences, as well as in the body of mathematics itself, is not an accident. The utility of polynomials is based largely on mathematical properties that, for all practical purposes, permit the replacement of much more complicated functions by polynomial functions in a host of situations. We shall enumerate some of these properties:

(a) Polynomial functions are among the simplest functions to manipulate formally. The sum, product, and composite of polynomial functions, the determination of slope and area, and the location of zeros and maxima and minima are all within the reach of elementary methods.

(b) Polynomial functions are among the simplest functions to evaluate. It is quite easy to find the value of $f(x)$, given

$$f : x \rightarrow a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

with a specific set of coefficients $a_0, a_1, \ldots, a_n$ and a specific number for x. Nothing more than multiplication and addition is involved, and the computation can be shortened by using the method of synthetic substitution.

The foregoing two properties of polynomial functions are those that make them valuable as replacements for more complicated functions.

(c) Frequently an experimental scientist makes a series of measurements, plots them as points, and then tries to find a reasonably simple continuous curve that will pass through these points. The graph of a polynomial function can always be used for this purpose, and because it has no sharp changes of direction, and only a limited number of ups and downs, it is in many ways the best curve for the purpose.

Thus, for the purpose of fitting a continuous graph to a finite number of points, we would prefer to work with polynomials and we need not look beyond the polynomials, as we shall prove. We can state the problem formally as follows:

Given n distinct numbers $x_1, x_2, \ldots, x_n$ and corresponding values $y_1, y_2, \ldots, y_n$ that a function is supposed to assume, it is possible to find a polynomial function of degree at most $n - 1$ whose graph contains the n points (x_i, y_i), $i = 1, \ldots, n$. You have already done this for $n = 2$: you found a linear or constant function whose graph contained two given points (x_1, y_1), (x_2, y_2), $x_2 \neq x_1$. If y_2 is also different from y_1, the result is a linear function; if $y_2 = y_1$, it is a constant function.

One way of doing this is to assume a polynomial of the stated form,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1},$$

and write the n equations

$$f(x_i) = y_i, \; i = 1, 2, \ldots, n.$$

This gives n linear equations in the n unknowns $a_0, a_1, \ldots, a_{n-1}$, and in these circumstances such a system will always have a solution.

Example A2-1a. Suppose that we want the graph of a function to pass through the points $(-2,2)$, $(1,3)$, $(2,-1)$, and $(4,1)$. We know that there is a polynomial graph of degree no greater than 3 which goes through these points. Assume, therefore,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Then, if the graph of f is to go through the given points, we must have $f(-2) = 2$, $f(1) = 3$, $f(2) = -1$, and $f(4) = 1$; that is,

$$\begin{aligned} a_0 - 2a_1 + 4a_2 - 8a_3 &= 2, \\ a_0 + a_1 + a_2 + a_3 &= 3, \\ a_0 + 2a_1 + 4a_2 + 8a_3 &= -1, \\ a_0 + 4a_1 + 16a_2 + 64a_3 &= 1. \end{aligned}$$

Solving these, we find $a_0 = \frac{20}{3}$, $a_1 = -\frac{31}{12}$, $a_2 = -\frac{37}{24}$, and $a_3 = \frac{11}{24}$. Hence

$$f(x) = \frac{1}{24}(160 - 62x - 37x^2 + 11x^3).$$

The labor of solving systems of linear equations such as these can be rather discouraging, especially if there are many equations. For this reason, various methods have been worked out for organizing and reducing the labor involved. One of the most important of these methods, called the Lagrange Interpolation Formula, is based on the following simple line of reasoning. We can easily write down a formula for a polynomial of degree $n - 1$ that is zero at $n - 1$ of the given x's.

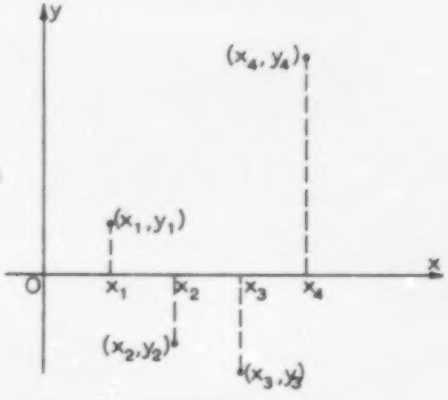

Figure A2-1a

A set of values to be taken on by a polynomial function. Suppose, for instance, that we have four points (x_1,y_1), (x_2,y_2), (x_3,y_3) and (x_4,y_4) as in Figure A2-1a. The polynomial

$$g_1(x) = C_1(x - x_2)(x - x_3)(x - x_4) \tag{1}$$

has zeros at x_2, x_3, and x_4. By proper choice of C_1, we can make $g_1(x_1) = y_1$. Let us do so! Take C_1 such that

$$y_1 = g_1(x_1) = C_1(x_1 - x_2)(x_1 - x_3)(x_1 - x_4),$$

that is, take

$$C_1 = \frac{y_1}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} . \tag{2}$$

If we substitute C_1 from (2) into (1), we get

$$g_1(x) = y_1 \cdot \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} . \tag{3}$$

If $y_1 \neq 0$, Equation (3) defines a polynomial of degree 3 that has the value y_1 at x_1 and is zero at x_2, x_3, and x_4. Similarly, one finds that

$$g_2(x) = y_2 \cdot \frac{(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} , \tag{4}$$

$$g_3(x) = y_3 \cdot \frac{(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)}, \tag{5}$$

and

$$g_4(x) = y_4 \cdot \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}, \tag{6}$$

are also polynomials, each having the property that it is zero at three of the four given values of x, and is the appropriate y at the fourth x. This is shown in the table below.

The Lagrange Interpolation Formula Illustrated.

Values of x	x_1	x_2	x_3	x_4
Corresponding y	y_1	y_2	y_3	y_4
Value of $g_1(x)$	y_1	0	0	0
Value of $g_2(x)$	0	y_2	0	0
Value of $g_3(x)$	0	0	y_3	0
Value of $g_4(x)$	0	0	0	y_4

If we form the sum

$$g(x) = g_1(x) + g_2(x) + g_3(x) + g_4(x), \tag{7}$$

then it is clear from the table that

$$\begin{aligned}
g(x_1) &= y_1 + 0 + 0 + 0 = y_1, \\
g(x_2) &= 0 + y_2 + 0 + 0 = y_2, \\
g(x_3) &= 0 + 0 + y_3 + 0 = y_3, \\
g(x_4) &= 0 + 0 + 0 + y_4 = y_4.
\end{aligned}$$

From Equations (3), (4), (5), and (6) it is also clear that g is a polynomial in x whose degree is at most 3. Hence Equation (7) tells us how to find a polynomial of degree ≤ 3, whose graph contains the given points.

Example A2-1b. Find a polynomial of degree at most 3 whose graph contains the points $(-1,2)$, $(0,0)$, $(2,-1)$, and $(4,2)$.

We find that

$$g_1(x) = 2\frac{(x - 0)(x - 2)(x - 4)}{(-1 - 0)(-1 - 2)(-1 - 4)} = \frac{2x(x - 2)(x - 4)}{-15},$$

$$g_2(x) = 0,$$

$$g_3(x) = -1\,\frac{(x + 1)(x - 0)(x - 4)}{(2 + 1)(2 - 0)(2 - 4)} = \frac{(x + 1)(x)(x - 4)}{12},$$

$$g_4(x) = 2\,\frac{(x + 1)(x - 0)(x - 2)}{(4 + 1)(4 - 0)(4 - 2)} = \frac{(x + 1)(x)(x - 2)}{20},$$

and

$$g(x) = -\frac{2}{15}\,x(x - 2)(x - 4) + \frac{1}{12}(x + 1)(x)(x - 4) + \frac{1}{20}(x + 1)(x)(x - 2)$$

$$= \frac{1}{2}(x^2 - 3x).$$

The right-hand sides of Equations (3), (4), (5), and (6) have the following structure:

$$g_i(x) = y_i\,\frac{N_i(x)}{D_i}\ ;\ i = 1, 2, 3, 4.$$

The numerator of the fraction is the product of all but one of the factors

$$(x - x_1),\ (x - x_2),\ (x - x_3),\ \text{and}\ (x - x_4),$$

and the missing factor is $(x - x_i)$. The denominator is the value of the numerator at $x = x_i$;

$$D_i = N_i(x_i).$$

This same structure would still hold if we had more (or fewer) points given.

(d) Instead of a finite set of points to which a simple continuous function is to be fitted, a mathematician is sometimes confronted with a continuous but very complicated function that he would like to approximate by a simpler function. Fortunately, there is an extremely powerful theorem of higher mathematics that enlarges the breadth of application of polynomials to this situation. In a sense this theorem permits the "fitting" of a polynomial graph to any continuous graph. In other words, any continuous function whatever can be approximated by a polynomial function over a finite interval of its domain, with preassigned accuracy. More specifically, if the function $x \rightarrow f(x)$ is continuous over $a \leq x \leq b$, and c is any positive number, there exists a polynomial function g such that

$$|f(x) - g(x)| < c \quad \text{for all } x \text{ in } a \leq x \leq b.$$

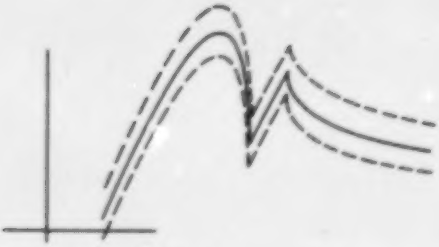

Figure A2-1c

A strip between $f(x) - c$ and $f(x) + c$.

This is known as the Weierstrass Approximation Theorem. The geometric interpretation of the theorem is indicated in Figure A2-1c The graph of f is a continuous curve, but it may have sharp corners or even infinitely many maxima and minima between $x = a$ and $x = b$. No polynomial graph behaves like that. But suppose that we introduce a strip, centered on the graph of f, extending between the graphs of the functions

$$x \to f(x) - c$$

and

$$x \to f(x) + c,$$

where c is any preassigned positive number, however small. Then the theorem guarantees that there is a polynomial function

$$g : x \to g(x),$$

whose graph on $a \leq x \leq b$ lies entirely inside this strip. This is the precise meaning of the statement: "Any continuous function whatever can be approximated by a polynomial function over a finite interval of its domain, with preassigned accuracy."

Exercises A2-1

1. Carry out the computations in Example A2-1a, above.
2. Simplify the expression for $g(x)$ in Example A2-1b, above.
3. Find a polynomial function of degree less than or equal to 2 whose graph contains the points $(-1,2)$, $(0,-1)$, $(2,3)$.
4. Find a polynomial function whose graph contains the points $(0,1)$, $(1,0)$, $(2,9)$, $(3,34)$, and $(1,6)$.

A2-2. Number of Zeros

One could get the impression that every polynomial function of degree $n > 0$ has exactly n zeros. This is not quite right; what we say in Section 1-9 is that every such function has at most n zeros. Let us exhibit a polynomial function for which the number of zeros is less than the degree. The quadratic function

$$f : x \to x^2 - 6x + 9 = (x - 3)^2$$

has only one zero, namely 3. But since the quadratic has two identical factors $x - 3$, we say that the zero 3 has multiplicity two.

We define the multiplicity of a zero r of a polynomial f to be the exponent of the highest power of $x - r$ that divides $f(x)$. That is, if

$$f(x) = (x - r)^k q(x), \quad k > 0,$$

where $q(x)$ is a polynomial, and if $x - r$ does not divide $q(x)$, then r is a zero of f of multiplicity k.

The proof of the general theorem about the number of zeros of a polynomial function depends on the fact that every such function has at least one zero. This fact, often referred to as Gauss's Theorem, is stated as follows:

> The Fundamental Theorem of Algebra. Every polynomial function of degree greater than zero has at least one zero, real or complex.

This is the simplest form of the Fundamental Theorem of Algebra. (As a matter of fact, the theorem is correct even if some or all of the coefficients of the polynomial are complex numbers.)

The first known proof of the theorem was published by the great German mathematician Carl Friedrich Gauss (1777 - 1855) in 1799. (Eric Temple Bell has written an interesting account of Gauss. See World of Mathematics, Simon and Schuster, 1956, Volume 1, pages 295-339, or E. T. Bell, Men of Mathematics, Simon and Schuster, 1937, pages 218-269.) The proof was contained in Gauss's doctoral dissertation, published when he was 22. A translation of his second proof (1816) is in A Source Book in Mathematics, by David Eugene Smith, McGraw-Hill Book Co., 1929, pages 292-310. Gauss gave a total of four different proofs of the theorem, the last in 1850. None of the proofs is sufficiently elementary to be given here. There is a proof in Birkhoff and MacLane, *A Survey of Modern Algebra*, Macmillan, 1953, pages 107-109. A proof is also given in L. E. Dickson, *New First Course in the Theory of Equations*, John Wiley and Sons, 1939.

We now state and prove the general theorem.

The General Form of the Fundamental Theorem of Algebra. Let f be a polynomial function of degree $n > 0$. Then f has at least one and at most n complex zeros, and the sum of the multiplicities of the zeros is exactly n.

From Section 1-9 we know that f has at least one zero, say r_1. Then (recall the Factor Theorem) there is a polynomial $q(x)$ of degree $n - 1$ such that

$$f(x) = (x - r_1)q(x). \tag{1}$$

If $n = 1$, q is of degree zero and we have finished. If $n > 1$, the degree of q is $n - 1$ and is positive. Then q has at least one zero r_2 (it could happen that $r_2 = r_1$) and

$$q(x) = (x - r_2)s(x), \tag{2}$$

where s is of degree $n - 2$. Combining (1) and (2) gives

$$f(x) = (x - r_1)(x - r_2)s(x). \tag{3}$$

If $n = 2$, then s in Equation (3) is of degree zero and we have finished. Otherwise, the process may be continued until we arrive at the final stage,

$$f(x) = (x - r_1)(x - r_2) \ldots (x - r_n)z(x), \tag{4}$$

where the degree of z is $n - n = 0$. Hence, $z(x)$ is a constant. Comparison of the expanded form of Equation (4) with the equivalent form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

shows that $z(x) = a_n \neq 0$. Hence,

$$f(x) = a_n(x - r_1)(x - r_2) \ldots (x - r_n). \tag{5}$$

Now, if we substitute any complex number r different from $r_1, r_2, \ldots, r_n$ in place of x in Equation (5), we get

$$f(r) = a_n(r - r_1)(r - r_2) \ldots (r - r_n).$$

Since every factor is different from zero, the product cannot be zero. Hence, no number except $r_1, r_2, \ldots, r_n$ is a zero of f, and f has at most n zeros.

Since it is possible that some of the r_i's may be equal, the number of zeros of f may be less than n. But Equation (5) shows that f has exactly n factors of the form $x - r_i$, and therefore the sum of the multiplicities of the zeros must be n.

Example A2-2a.

$$f : x \to x^5 + x^4 - 5x^3 - x^2 + 8x - 4$$

has zeros of multiplicity greater than one. Find the zeros and indicate the multiplicity of each.

Since the coefficient of the term of highest degree is 1, we know that any rational zeros of f must be integers that are factors of 4. (Refer to Section 1-7.) Using synthetic substitution and the polynomial of reduced degree obtained each time a zero is found, we discover that 1 is a zero of multiplicity three and -2 is a zero of multiplicity two. Note that the sum of the multiplicities is five, which is also the degree of the given polynomial.

It may be helpful to show a practical way for putting down the synthetic substitutions by which we obtained the zeros and their multiplicities. This is done in Table A2-2.

Table A2-2

Finding the Zeros of $f : x \to x^5 + x^4 - 5x^3 - x^2 + 8x - 4$

$$\begin{array}{cccccc|l}
1 & 1 & -5 & -1 & 8 & -4 & \underline{1} \\
 & 1 & 2 & -3 & -4 & 4 & \\
\hline
1 & 2 & -3 & -4 & 4 & 0 & \\
\hline
1 & 2 & -3 & -4 & 4 & & \underline{1} \\
 & 1 & 3 & 0 & -4 & & \\
\hline
1 & 3 & 0 & -4 & 0 & & \\
\hline
1 & 3 & 0 & -4 & & & \underline{1} \\
 & 1 & 4 & 4 & & & \\
\hline
1 & 4 & 4 & 0 & & & \\
\hline
\end{array}$$

1 is a zero of f of multiplicity three.

The entri[illegible] 2, -3, -4, 4 in the third row, 1, 3, 0, -4 in the sixth row, and 1, 4, 4 in the last row are coefficients of polynomials of degree four, three, and two, respectively. The quadratic function $x \to x^2 + 4x + 4$ has -2 as a zero of multiplicity two since $x^2 + 4x + 4 = (x + 2)^2$.

Thus, the zeros of f are 1 (of multiplicity three) and -2 (of multiplicity two).

The graph of f is shown in Figure A2-2a in order to give you some idea of its shape in the neighborhood of the zeros -2 and 1 (points A and B).

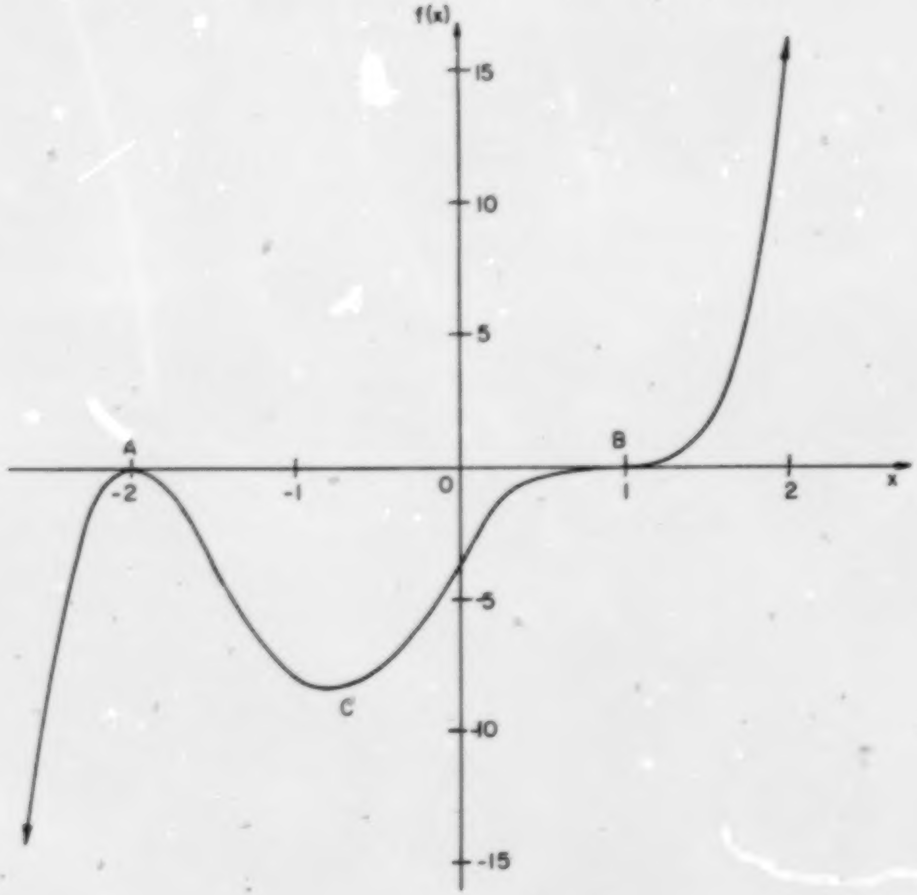

Figure A2-2a

Graph of $f : x \to x^5 + x^4 - 5x^3 - x^2 + 8x - 4$.

The Fundamental Theorem of Algebra implies that the range of any nonconstant polynomial function includes zero when its domain is the set of all complex numbers. The range does not always include zero when the domain is the set of real numbers. For example, if

$$f : x \to y = x^2 + 1, \quad x \in R,$$

then the range of f is the set

$$\{y : y \geq 1\}.$$

When the domain of f is the set of complex numbers, and the degree of f is > 0, then its range is also the set of all complex numbers. For, suppose that f is a polynomial of degree $n > 0$ and $a + ib$ is any complex number. Then the equation

$$f(x) = a + ib$$

is equivalent to

$$f(x) - a - ib = 0. \tag{6}$$

This is a polynomial equation of degree n; hence, by the Fundamental Theorem of Algebra, Equation (6) has a solution. That is, there exists at least one complex number x that is mapped by f into $a + ib$:

$$f(x) = a + ib.$$

Moreover, there may be as many as n different numbers in the domain that map into $a + ib$, and the sum of the multiplicities of the solutions of (6) will be exactly n.

The Fundamental Theorem does not tell us how to find even one of the zeros of f. It just guarantees that they exist. The general problem of finding a complex zero of an arbitrary polynomial is quite difficult. In the 1930's the Bell Telephone Laboratories built a machine, the Isograph, for solving such problems when the degree is 10 or less. See The Isograph -- A Mechanical Root-Finder, by R. L. Dietzold, Bell Labs Record 16, December, 1937, page 130. Nowadays, electronic computers are used to do this job, and many others. Numerous applications of computers in science and industry are discussed in a series of articles in the book The Computing Laboratory in the University, University of Wisconsin Press, Madison, Wisconsin, 1957, edited by Preston C. Hammer.

The following quotation is taken from a book called Mathematics and Computers, by George R. Stibitz and Jules A. Larrivee, McGraw-Hill Book Co., New York, 1957, page 37:

"There is an interesting use for the roots of the 'characteristic equation' of a vibrating system in the dynamics of electromagnetic and mechanical systems where many of the properties of amplifiers, filters, servos, airfoils, and other devices must be determined. If any one of the complex roots of characteristic equation for a system has a positive real part, the system will be unstable: amplifiers will howl, servos will oscillate uncontrollably, and bridges will collapse under the stresses exerted by the winds. The prediction of such behavior is of great importance to designers of the amplifiers that boost your voice as it crosses the country over telephone lines, and the servos that point guns at an attacking plane."

Exercises A2-2

1. Assume that the equations given below are the characteristic equations of some mechanical or electrical system. According to the quotation from Stibitz and Larrivee, are the systems stable or unstable?

   (a) $x^3 - x^2 + 2 = 0$,

   (b) $x^3 - 3x^2 + 4x - 2 = 0$,

   (c) $x^3 + 3x^2 + 4x + 2 = 0$,

   (d) $x^3 + x^2 - 2 = 0$,

   (e) $x^3 + 6x^2 + 13x + 10 = 0$.

2. The following equations have multiple roots. Find them and, in each case, show that the sum of the multiplicities of the roots equals the degree of the polynomial.

   (a) $x^3 - 3x - 2 = 0$,

   (b) $x^3 - 3x + 2 = 0$,

   (c) $x^4 + 5x^3 + 9x^2 + 7x + 2 = 0$.

3. Find the roots and their multiplicities of each of the following equations. Compare the solution sets of the two equations.

   (a) $x^5 + 4x^4 + x^3 - 10x^2 - 4x + 8 = 0$

   (b) $x^5 + x^4 - 5x^3 - x^2 + 8x - 4 = 0$

4. A number system is said to be algebraically closed if, and only if, every polynomial equation of degree > 0, with coefficients in that system, has a solution in that system. Which of the following number systems are, and which are not, algebraically closed? Give reasons for your answers.

   (a) The integers: ..., -2, -1, 0, 1, 2, 3, ...

   (b) The rational numbers.

   (c) The real numbers.

   (d) The pure imaginary numbers bi.

   (e) The complex numbers.

5. You may have heard that it was necessary for mathematicians to invent $\sqrt{-1}$ and other complex numbers in order to solve some quadratic equations. Do you suppose that they needed to invent something that might be called "super-complex" numbers to express such things as $\sqrt[4]{-1}$, $\sqrt[6]{-1}$, and so on? Give reasons for your answers.

A2-3. *Complex Zeros*

We know that a quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0, \tag{1}$$

has roots given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\ . \tag{2}$$

The coefficients a, b, and c in (1) are here assumed to be real numbers. The quantity under the radical in (2) is called the *discriminant*. Its sign determines the nature of the roots of (1). The roots are

(a) real and unequal if $b^2 - 4ac > 0$,

(b) real and equal if $b^2 - 4ac = 0$,

(c) imaginary if $b^2 - 4ac < 0$.

Example A2-3a. What are the roots of $x^2 + x + 1 = 0$?

The roots are

$$\frac{-1 + i\sqrt{3}}{2}, \quad \frac{-1 - i\sqrt{3}}{2}\ .$$

We notice that these roots are *complex conjugates*; that is, they have the form $u + iv$ and $u - iv$, where u and v are real. In this example, $u = -\frac{1}{2}$ and $v = \frac{\sqrt{3}}{2}$.

Is it just a coincidence that these roots are complex conjugates? Let us look at (2), and suppose that a, b, and c are real numbers and that the discriminant is negative, say $-d^2$. Then the roots of $ax^2 + bx + c = 0$ are $-\frac{b}{2a} + i\left(\frac{d}{2a}\right)$ and $-\frac{b}{2a} - i\left(\frac{d}{2a}\right)$. These are complex conjugates. Thus, if a, b, and c are real and if the roots of (1) are imaginary, then these roots are complex conjugates. This is true of polynomials of any degree, as we shall now prove. (In the following theorem, the letters a and b represent the real and imaginary parts of a complex root of an equation of any degree, and do not refer to the coefficients in a quadratic expression.)

Complex-conjugates Theorem. If $f(x)$ is a polynomial with real coefficients, and if $a + ib$ is a complex root of $f(x) = 0$, with imaginary part $b \neq 0$, then $a - ib$ is also a root.

(Another way of saying this is that if $f(a + ib) = 0$, with a and b real and $b \neq 0$, then $f(a - ib) = 0$.)

We give two proofs of this result.

First Proof. The key to this proof is the use of the quadratic polynomial that is the product of $x - (a + ib)$ and $x - (a - ib)$. We show that it divides $f(x)$. We can then conclude that $f(a - ib) = 0$, and we have completed the proof.

Thus, let

$$\begin{aligned} p(x) &= [x - (a + ib)][x - (a - ib)] \\ &= [(x - a) - ib][(x - a) + ib] \\ &= (x - a)^2 + b^2. \end{aligned} \tag{3}$$

Note that $p(x)$ is a quadratic polynomial with real coefficients. Now when a polynomial is divided by a quadratic, a remainder of degree less than 2 is obtained. Hence, if $f(x)$ is divided by $p(x)$, we get a polynomial quotient $q(x)$ and a remainder $r(x) = hx + k$, possibly of degree 1 (but no greater), where h, k, and all the coefficients of $q(x)$ are real. Thus,

$$f(x) = p(x) \cdot q(x) + hx + k. \tag{4}$$

This is an identity in x. By hypothesis, $f(a + ib) = 0$, and from Equation (3), $p(a + ib) = 0$. Therefore, if we substitute $a + ib$ for x in Equation (4), we get

$$0 = 0 + ha + ihb + k.$$

Since real and imaginary parts must both be 0, we have

$$ha + k = 0, \tag{5}$$

and

$$hb = 0. \tag{6}$$

Since $b \neq 0$ (by hypothesis), Equation (6) requires that $h = 0$. Then Equation (5) gives $k = 0$. Therefore, the remainder $hx + k$ in Equation (4) is zero, and

$$f(x) = p(x) \cdot q(x). \tag{7}$$

Since $p(a - ib) = 0$ by Equation (3), it follows from Equation (7) that

$$f(a - ib) = 0.$$

Second Proof. Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \tag{8}$$

and suppose that $f(a + ib) = 0$. When we substitute $a + ib$ for x in Equation (8), we can expand $(a + ib)^2$, $(a + ib)^3$, and so on, by the Binomial Theorem. We can prove the complex-conjugates theorem, however, without actually carrying out all of these expansions, if we observe how the terms behave. Consider the first few powers of $a + ib$:

$$\begin{aligned}
(a + ib)^1 &= a + ib, \\
(a + ib)^2 &= a^2 + 2aib + i^2b^2 \\
&= (a^2 - b^2) + i(2ab), \\
(a + ib)^3 &= a^3 + 3a^2ib + 3a(i^2b^2) + i^3b^3 \\
&= (a^3 - 3ab^2) + i(3a^2b - b^3).
\end{aligned}$$

Now observe where b occurs in the above expanded forms. In the real parts, b either does not occur at all, or it occurs only to even powers. In the imaginary parts, b always occurs to odd powers. This follows from the fact that all even powers of i are real and all odd powers are imaginary. If we change the sign of b, we therefore leave the real part unchanged and change the sign of the imaginary part. Thus, if $f(a + ib) = u + iv$, then $f(a - ib) = u - iv$. But by hypothesis,

$$f(a + ib) = 0,$$

so that

$$u + iv = 0,$$

and therefore

$$u = v = 0.$$

Hence

$$f(a - ib) = 0.$$

Example A2-4a. What is the degree of a polynomial function f of minimum degree if $2 + i$, 1, and $3 - 2i$ are zeros of f?

If it is not required that the coefficients of the polynomial be real, then we may take

$$f(x) = [x - (2 + i)][x - 1][x - (3 - 2i)]$$
$$= x^3 + (-6 + i)x^2 + (13 - 2i)x + (-8 + i).$$

In this case, the degree of f is 3. No polynomial function of lower degree can have 3 zeros, so 3 is the answer. However, if it is required that the coefficients of $f(x)$ be real, then the answer to the question is 5. For then the conjugates of $2 + i$ and $3 - 2i$ must also be zeros of f. No polynomial function of degree less than 5 can have the 5 zeros

(9) $$2 + i,\ 2 - i,\ 1,\ 3 - 2i,\ 3 + 2i.$$

But

(10) $$[x - (2 + i)][x - (2 - i)][x - 1][x - (3 - 2i)][x - (3 + 2i)]$$

is a polynomial of degree 5, with real coefficients, that does have the numbers listed in (9) as its zeros.

Exercises A2-3

1. Multiply the factors in (10) above to show that the expression does have real coefficients. What is the coefficient of x^4 in your answer? What is the constant term? Compare these with the sum and the product of the zeros listed in (9).

2. Write a polynomial function of minimum degree that has $2 + 3i$ as a zero,
   (a) if imaginary coefficients are allowed,
   (b) if the coefficients must be real.

3. Find all roots of the following equations:

   (a) $x^3 - 1 = 0$

   (b) $x^3 + 1 = 0$

   (c) $x^3 - x^2 + 2x = 8$

   (d) $x^4 + 5x^2 + 4 = 0$

   (e) $x^4 - 2x^3 + 10x^2 - 18x + 9 = 0$

   (f) $x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1 = 0$

   (g) $x^6 - 2x^5 + 3x^4 - 4x^3 + 3x^2 - 2x + 1 = 0$

4. What is the degree of the polynomial equation of minimum degree with real coefficients having $2 + i$, $-2 + i$, $2 - i$, $3 + i$, $-3 + i$ as roots?

5. Consider the set of numbers of the form $a + b\sqrt{2}$, where a and b are rational. Then $a - b\sqrt{2}$ is called the conjugate surd of $a + b\sqrt{2}$. Prove the following theorem on conjugate surds:

   If $f(x)$ is a polynomial with rational coefficients, and if $a + b\sqrt{2}$ is a root of $f(x) = 0$, then $a - b\sqrt{2}$ is also a root. (Note that if $u = v\sqrt{2} = 0$, and u and v are rational, then $u = v = 0$. Otherwise, we could solve for $\sqrt{2} = -\frac{u}{v}$, the quotient of two rational numbers. But we know that $\sqrt{2}$ is irrational.)

6. Find a polynomial with rational coefficients and minimum degree having $3 + 2\sqrt{2}$ as a zero.

7. State and prove a theorem similar to that in Exercise 5 above for numbers of the form $a + b\sqrt{3}$. Is there a comparable theorem about roots of the form $a + b\sqrt{4}$? Give reasons for your answers.

8. Write a polynomial function of minimum degree that has -1 and $3 - 2\sqrt{3}$ as zeros, if

   (a) irrational coefficients are allowed;

   (b) the coefficients must be rational.

9. Find a polynomial of minimum degree with rational coefficients having $\sqrt{3} + \sqrt{2}$ as a zero.

10. What is the degree of a polynomial of minimum degree with (a) real, and (b) rational coefficients having

    (1) $i + \sqrt{2}$ as a zero?

    (2) $1 + i\sqrt{2}$ as a zero?

    (3) $\sqrt{2} + i\sqrt{3}$ as a zero?